

THE WEAK IDEAL PROPERTY AND TOPOLOGICAL DIMENSION ZERO

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ABSTRACT. Following up on previous work, we prove a number of results for C^* -algebras with the weak ideal property or topological dimension zero, and some results for C^* -algebras with related properties. Some of the more important results include:

- The weak ideal property implies topological dimension zero.
- For a separable C^* -algebra A , topological dimension zero is equivalent to $\text{RR}(\mathcal{O}_2 \otimes A) = 0$, to $D \otimes A$ having the ideal property for some (or any) Kirchberg algebra D , and to A being residually hereditarily in the class of all C^* -algebras B such that $\mathcal{O}_\infty \otimes B$ contains a nonzero projection.
- Extending the known result for \mathbb{Z}_2 , the classes of C^* -algebras with residual (SP), which are residually hereditarily (properly) infinite, or which are purely infinite and have the ideal property, are closed under crossed products by arbitrary actions of abelian 2-groups.
- If A and B are separable, one of them is exact, A has the ideal property, and B has the weak ideal property, then $A \otimes_{\min} B$ has the weak ideal property.
- If X is a totally disconnected locally compact Hausdorff space and A is a $C_0(X)$ -algebra all of whose fibers have one of the weak ideal property, topological dimension zero, residual (SP), or the combination of pure infiniteness and the ideal property, then A also has the corresponding property (for topological dimension zero, provided A is separable).
- Topological dimension zero, the weak ideal property, and the ideal property are all equivalent for a substantial class of separable C^* -algebras including all separable locally AH algebras.
- The weak ideal property does not imply the ideal property for separable \mathbb{Z} -stable C^* -algebras.

We give other related results, as well as counterexamples to several other statements one might hope for.

The weak ideal property (recalled in Definition 1.3 below) was introduced in [29]; it is the property for which there are good permanence results (see Section 8 of [29]) which seems to be closest to the ideal property. (The ideal property fails to pass to extensions, by Theorem 5.1 of [22], to corners, by Example 2.8 of [28], and to fixed point algebras under actions of \mathbb{Z}_2 , by Example 2.7 of [28]. The weak ideal property does all of these.) Topological dimension zero was introduced in [5]; it is a non-Hausdorff version of total disconnectedness of the primitive ideal space of

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a C^* -algebra. These two properties are related, although not identical, and the purpose of this paper is to study them and their connections further. Some of our results also involve the ideal property, real rank zero, and several forms of pure infiniteness.

For simple C^* -algebras, the appropriate regularity properties (real rank zero, Property (SP), Z -stability, strict comparison, pure infiniteness, etc.) are fairly well understood. For nonsimple C^* -algebras, there are more such properties, they are less well understood, and it is not clear which of them are the “right” ones to consider. This paper is a contribution towards a better understanding of some of these properties.

Even though it is not yet clear what the right regularity properties in the nonsimple case are, the properties we consider (topological dimension zero, the weak ideal property, and the ideal property) have at least proved to be valuable. For example, if X is the primitive ideal space of a separable C^* -algebra, then X has topological dimension zero if and only if X is the primitive ideal space of an AF algebra. (See Section 3 and the theorem in Section 5 of [3].) If A is a separable purely infinite C^* -algebra, then A has real rank zero if and only if A has topological dimension zero and is K_0 -liftable (Theorem 4.2 of [31]).

Turning to the ideal property (every ideal is generated, as an ideal, by its projections), we consider AH algebras (in the sense of [21]: the spaces used are connected finite complexes) with the ideal property and with slow dimension growth. Such algebras have stable rank one (Theorem 4.1 of [21]), and can be classified up to shape equivalence by a K -theoretic invariant (Theorem 2.15 of [21]). If A is such an algebra and $K_*(A)$ is torsion free, then A is an AT algebra, that is, a direct limit of finite direct sums of matrix algebras over $C(S^1)$ (Theorem 3.6 of [10]; this paper uses a less restrictive definition of AH algebra). The stable rank one and AT algebra results fail without the ideal property. (Counterexamples are easy, but too long for the introduction; we present them at the beginning of Section 6.) Also, a separable purely infinite C^* -algebra has the ideal property if and only if it has topological dimension zero (Proposition 2.11 of [31].)

The weak ideal property is much more recent. As noted above, it has better permanence properties than the ideal property. Moreover, under the hypotheses of the theorems above for the ideal property, the weak ideal property actually implies the ideal property. (See Theorem 6.12 and Theorem 1.10.)

We next describe our results.

We prove in Section 1 that the weak ideal property implies topological dimension zero in complete generality. For separable C^* -algebras which are purely infinite in the sense of [16], it is equivalent to the ideal property and to topological dimension zero. A general separable C^* -algebra A has topological dimension zero if and only if $\mathcal{O}_2 \otimes A$ has real rank zero; this is also equivalent to $D \otimes A$ having the ideal property for some (or any) Kirchberg algebra D . We rule out by example other results in this direction that one might hope for. Topological dimension zero, at least for separable C^* -algebras, is also equivalent to a property of the sort considered in [29]. That is, there is an upwards directed class \mathcal{C} such that a separable C^* -algebra A has topological dimension zero if and only if A is residually hereditarily in \mathcal{C} . (See the end of the introduction for other examples of this kind of property.)

In Section 2, we improve the closure properties under crossed products of the class of C^* -algebras residually hereditarily in a class \mathcal{C} by replacing an arbitrary

action of \mathbb{Z}_2 with an arbitrary action of a finite abelian 2-group. (See Corollary 2.3.) This refinement was overlooked in [29]. It applies to residual hereditary (proper) infiniteness as well as to residual (SP) and to the combination of pure infiniteness and the ideal property. For the weak ideal property and for topological dimension zero, better results are already known (Corollary 8.10 of [29] and Theorem 3.17 of [28]). However, for topological dimension zero, in the separable case we remove the technical hypothesis in Theorem 3.14 of [28], and show that if a finite group acts on a separable C^* -algebra A and the fixed point algebra has topological dimension zero, then A has topological dimension zero.

Section 3 considers minimal tensor products. For a tensor product to have the weak ideal property or topological dimension zero, it is usually necessary that both tensor factors have the corresponding property. In the separable case and with one factor exact, this is sufficient for topological dimension zero. We show by example that this result fails without the exactness hypothesis. For the weak ideal property, we get only partial results: if both factors are separable, one is exact, and one actually has the ideal property, or if one factor is exact and one factor has finite or Hausdorff primitive ideal space, then the tensor product has the weak ideal property.

Proceeding to a $C_0(X)$ -algebra A , we show that if X is totally disconnected and the fibers all have the weak ideal property, topological dimension zero, residual (SP), or the combination of pure infiniteness and the ideal property, then A also has the corresponding property (for topological dimension zero, provided A is separable). This result is the analog for these properties of Theorem 2.1 of [25] (for real rank zero) and Theorem 2.1 of [26] (for the ideal property), but we do not assume that the $C_0(X)$ -algebra is continuous. If A is a separable continuous $C_0(X)$ -algebra with nonzero fibers and X is second countable, then total disconnectedness of X is also necessary. This is in Section 4. In the short Section 5, we consider locally trivial $C_0(X)$ -algebras with fibers which are strongly purely infinite in the sense of Definition 5.1 of [17], and show (slightly generalizing the known result for $C_0(X, B)$) that A is again strongly purely infinite. In particular, this applies if the fibers are separable, purely infinite, and have topological dimension zero.

Section 6 gives a substantial class of C^* -algebras for which the ideal property, the weak ideal property, and topological dimension zero are all equivalent. This class includes all separable locally AH algebras (using a somewhat restrictive definition: the spaces involved must have only finitely many connected components and the maps must be injective). It also includes further generalizations of separable AH algebras, such as separable LS algebras. We also prove that the weak ideal property implies the ideal property for stable C^* -algebras A such that $\text{Prim}(A)$ is Hausdorff. However, we show by example that there is a Z -stable C^* -algebra with just one nontrivial ideal which has the weak ideal property but not the ideal property.

Ideals in C^* -algebras are assumed to be closed and two sided. We write \mathbb{Z}_n for $\mathbb{Z}/n\mathbb{Z}$, since the p -adic integers will not appear. If $\alpha: G \rightarrow \text{Aut}(A)$ is an action of a group G on a C^* -algebra A , then A^α denotes the fixed point algebra.

Because of the role they play in this paper, we recall the following definitions from [29].

Definition 0.1 (Definition 5.1 of [29]). Let \mathcal{C} be a class of C^* -algebras. We say that \mathcal{C} is *upwards directed* if whenever A is a C^* -algebra which contains a subalgebra isomorphic to an algebra in \mathcal{C} , then $A \in \mathcal{C}$.

Definition 0.2 (Definition 5.2 of [29]). Let \mathcal{C} be an upwards directed class of C^* -algebras, and let A be a C^* -algebra.

- (1) We say that A is *hereditarily in \mathcal{C}* if every nonzero hereditary subalgebra of A is in \mathcal{C} .
- (2) We say that A is *residually hereditarily in \mathcal{C}* if A/I is hereditarily in \mathcal{C} for every ideal $I \subset A$ with $I \neq A$.

Section 5 of [29] gives permanence properties for a general condition defined this way. We recall the conditions of this type considered in [29], and add one more to be proved here.

- (1) Let \mathcal{C} be the class of all C^* -algebras which contain an infinite projection. Then \mathcal{C} is upwards directed (clear) and a C^* -algebra A is purely infinite and has the ideal property if and only if A is residually hereditarily in \mathcal{C} . See the equivalence of conditions (ii) and (iv) of Proposition 2.11 of [31] (valid, as shown there, even when A is not separable).
- (2) Let \mathcal{C} be the class of all C^* -algebras which contain an infinite element. Then \mathcal{C} is upwards directed (clear) and a C^* -algebra A is (residually) hereditarily infinite (Definition 6.1 of [29]) if and only if A is residually hereditarily in \mathcal{C} . (See Corollary 6.5 of [29]. We should point out that, by Lemma 2.2(iii) of [16], if D is a C^* -algebra, $B \subset D$ is a hereditary subalgebra, and a and b are positive elements of B such that a is Cuntz subequivalent to b relative to D , then a is Cuntz subequivalent to b relative to B .)
- (3) Let \mathcal{C} be the class of all C^* -algebras which contain a properly infinite element. Then \mathcal{C} is upwards directed (clear) and a C^* -algebra A is (residually) hereditarily properly infinite (Definition 6.2 of [29]) if and only if A is residually hereditarily in \mathcal{C} . (Lemma 2.2(iii) of [16] plays the same role here as in (2).)
- (4) Let \mathcal{C} be the class of all C^* -algebras which contain a nonzero projection. Then \mathcal{C} is upwards directed (clear). A C^* -algebra A has Property (SP) if and only if A is hereditarily in \mathcal{C} , and has residual (SP) (Definition 7.1 of [29]) if and only if A is residually hereditarily in \mathcal{C} . (Both statements are clear. Residual (SP) appears, without the name, as a hypothesis in the discussion after Proposition 4.18 of [17].)
- (5) Let \mathcal{C} be the class of all C^* -algebras B such that $K \otimes B$ contains a nonzero projection. Then \mathcal{C} is upwards directed (clear) and a C^* -algebra A has the weak ideal property (Definition 8.1 of [29]; recalled in Definition 1.3 below) if and only if A is residually hereditarily in \mathcal{C} . (This is shown at the beginning of the proof of Theorem 8.5 of [29].)
- (6) Let \mathcal{C} be the class of all C^* -algebras B such that $\mathcal{O}_2 \otimes B$ contains a nonzero projection. Then \mathcal{C} is upwards directed. (This is clear.) A separable C^* -algebra A has topological dimension zero if and only if A is residually hereditarily in \mathcal{C} . (This will be proved in Theorem 1.10 below.)

1. TOPOLOGICAL DIMENSION ZERO

In this section, we prove (Theorem 1.8) that the weak ideal property implies topological dimension zero for general C^* -algebras. We then give characterizations of topological dimension zero for separable C^* -algebras (Theorem 1.10) and purely infinite separable C^* -algebras (Theorem 1.9), in terms of other properties of the

algebra, in terms of properties of their tensor products with suitable Kirchberg algebras, and (for general separable C*-algebras) of the form of being residually hereditarily in suitable upwards directed classes. We also give two related counterexamples. In particular, there is a separable purely infinite unital nuclear C*-algebra A with one nontrivial ideal such that $\mathcal{O}_2 \otimes A \cong A$ and $\text{RR}(A) = 0$, and an action $\alpha: \mathbb{Z}_2 \rightarrow \text{Aut}(A)$, such that $\text{RR}(C^*(\mathbb{Z}_2, A, \alpha)) \neq 0$.

We recall two definitions from [28]. We call a not necessarily Hausdorff space *locally compact* if the compact (but not necessarily closed) neighborhoods of every point $x \in X$ form a neighborhood base at x .

Definition 1.1 (Remark 2.5(vi) of [5]; Definition 3.2 of [28]). Let X be a locally compact but not necessarily Hausdorff topological space. We say that X has *topological dimension zero* if for every $x \in X$ and every open set $U \subset X$ such that $x \in U$, there exists a compact open (but not necessarily closed) subset $Y \subset X$ such that $x \in Y \subset U$. (Equivalently, X has a base for its topology consisting of subsets which are compact and open, but not necessarily closed.) We further say that a C*-algebra A has *topological dimension zero* if $\text{Prim}(A)$ has topological dimension zero.

Definition 1.2 (Definition 3.4 of [28]). Let X be a not necessarily Hausdorff topological space. A *compact open exhaustion* of X is an increasing net $(Y_\lambda)_{\lambda \in \Lambda}$ of compact open subsets $Y_\lambda \subset X$ such that $X = \bigcup_{\lambda \in \Lambda} Y_\lambda$.

We further recall (Lemma 3.10 of [28]; see Definition 3.9 of [28] or page 53 of [31] for the original definition) that if A is a C*-algebra and $I \subset A$ is an ideal, then I is compact if and only if $\text{Prim}(I)$ is a compact open (but not necessarily closed) subset of $\text{Prim}(A)$.

Finally, we recall the definition of the weak ideal property.

Definition 1.3 (Definition 8.1 of [29]). Let A be a C*-algebra. We say that A has the *weak ideal property* if whenever $I \subset J \subset K \otimes A$ are ideals in $K \otimes A$ such that $I \neq J$, then J/I contains a nonzero projection.

Lemma 1.4. Let A be a C*-algebra with the weak ideal property. Let $I_1 \subset I_2 \subset A$ be ideals with $I_1 \neq I_2$. Then there exists an ideal $J \subset A$ with $I_1 \subsetneq J$ and such that $K \otimes (J/I_1)$ is generated as an ideal by a single nonzero projection.

Proof. Since A has the weak ideal property and $K \otimes (I_2/I_1) \neq 0$, there is a nonzero projection $e \in K \otimes (I_2/I_1)$. Let $I \subset K \otimes (I_2/I_1)$ be the ideal generated by e . Then there is an ideal $J \subset A$ with $I_1 \subset J \subset I_2$ such that $I = K \otimes (J/I_1)$. Since $J/I_1 \neq 0$, it follows that $J \neq I_1$. \square

Lemma 1.5. Let A be a C*-algebra, let $F \subset A$ be a finite set of projections, and let $I \subset A$ be the ideal generated by F . Then $\text{Prim}(I)$ is a compact open subset of $\text{Prim}(A)$.

Proof. This can be shown by using the same argument as in (iii) implies (i) in the proof of Proposition 2.7 of [31]. However, we can give a more direct proof (not involving the Pedersen ideal). As there, we prove that I is compact (as recalled after Definition 1.2). So let $(I_\lambda)_{\lambda \in \Lambda}$ be an increasing net of ideals in A such that $\bigcup_{\lambda \in \Lambda} I_\lambda = I$. Standard functional calculus arguments produce $\varepsilon > 0$ such that whenever B is a C*-algebra, $C \subset B$ is a subalgebra, and $p \in B$ is a projection such that $\text{dist}(p, C) < \varepsilon$, then there is a projection $q \in C$ such that $\|q - p\| < 1$, and

in particular q is Murray-von Neumann equivalent q . Write $F = \{p_1, p_2, \dots, p_n\}$. Choose $\lambda \in \Lambda$ such that $\text{dist}(p_j, I_\lambda) < \varepsilon$ for $j = 1, 2, \dots, n$. Let $q_1, q_2, \dots, q_n \in I_\lambda$ be projections obtained from the choice of ε . Then there are partial isometries $s_1, s_2, \dots, s_n \in A$ such that $p_j = s_j q_j s_j^*$ for $j = 1, 2, \dots, n$. So $p_1, p_2, \dots, p_n \in I_\lambda$, whence $I_\lambda = I$. This completes the proof. \square

Lemma 1.6. Let A be a C^* -algebra, and let $I \subset A$ be an ideal. Suppose that there is a collection $(I_\lambda)_{\lambda \in \Lambda}$ (not necessarily a net) of ideals in A such that I is the ideal generated by $\bigcup_{\lambda \in \Lambda} I_\lambda$ and such that $\text{Prim}(I_\lambda)$ has a compact open exhaustion (as in Definition 1.2) for every $\lambda \in \Lambda$. Then $\text{Prim}(I)$ has a compact open exhaustion.

Proof. It is easily checked that a union of open sets with compact open exhaustions also has a compact open exhaustion. \square

Proposition 1.7. Let A be a C^* -algebra. Then there is a largest ideal $I \subset A$ such that $\text{Prim}(I)$ has a compact open exhaustion.

Proof. Let I be the closure of the union of all ideals $J \subset A$ such that $\text{Prim}(J)$ has a compact open exhaustion. Then $\text{Prim}(I)$ has a compact open exhaustion by Lemma 1.6. \square

Theorem 1.8. Let A be a C^* -algebra with the weak ideal property. Then A has topological dimension zero.

Proof. We will show that for every ideal $I \subset A$, the subset $\text{Prim}(I)$ has a compact open exhaustion. The desired conclusion will then follow from Lemma 3.6 of [28].

So let $I \subset A$ be an ideal. By Proposition 1.7, there is a largest ideal $J \subset I$ such that $\text{Prim}(J)$ has a compact open exhaustion. We prove that $J = I$. Suppose not. Use Lemma 1.4 to find an ideal $N \subset I$ with $J \subsetneq N$ and such that $K \otimes (N/J)$ is generated by one nonzero projection. Then $\text{Prim}(K \otimes (N/J))$ is a compact open subset of $\text{Prim}(K \otimes (I/J))$ by Lemma 1.5. So $\text{Prim}(N/J)$ is a compact open subset of $\text{Prim}(I/J)$. Since $\text{Prim}(J)$ has a compact open exhaustion, we can apply Lemma 3.7 of [28] (taking $U = \text{Prim}(J)$) to deduce that $\text{Prim}(N)$ has a compact open exhaustion. Since $J \subsetneq N$, we have a contradiction. Thus $J = I$, and $\text{Prim}(I)$ has a compact open exhaustion. \square

The list of equivalent conditions in the next theorem extends the list in Corollary 4.3 of [31], by adding condition (5). As discussed in the introduction, this condition is better behaved than the related condition (4).

Theorem 1.9. Let A be a separable C^* -algebra which is purely infinite in the sense of Definition 4.1 of [16]. Then the following are equivalent:

- (1) $\mathcal{O}_2 \otimes A$ has real rank zero.
- (2) $\mathcal{O}_2 \otimes A$ has the ideal property.
- (3) A has topological dimension zero.
- (4) A has the ideal property.
- (5) A has the weak ideal property.

Proof. The equivalence of conditions (1), (2), (3), and (4) is Corollary 4.3 of [31]. That (4) implies (5) is trivial. That (5) implies (3) is Theorem 1.8. \square

We presume that Theorem 1.9 holds without separability. However, some of the results used in the proof of Corollary 4.3 of [31] are only known in the separable case, and it seems likely to require some work to generalize them.

Recall that a Kirchberg algebra is a simple separable nuclear purely infinite C^* -algebra.

Theorem 1.10. Let A be a separable C^* -algebra. Then the following are equivalent:

- (1) A has topological dimension zero.
- (2) $\mathcal{O}_2 \otimes A$ has real rank zero.
- (3) $\mathcal{O}_2 \otimes A$ has the ideal property.
- (4) $\mathcal{O}_2 \otimes A$ has the weak ideal property.
- (5) $\mathcal{O}_\infty \otimes A$ has the ideal property.
- (6) $\mathcal{O}_\infty \otimes A$ has the weak ideal property.
- (7) There exists a Kirchberg algebra D such that $D \otimes A$ has the weak ideal property.
- (8) For every Kirchberg algebra D , the algebra $D \otimes A$ has the ideal property.
- (9) A is residually hereditarily in the class of all C^* -algebras B such that $\mathcal{O}_2 \otimes B$ contains a nonzero projection.
- (10) A is residually hereditarily in the class of all C^* -algebras B such that $K \otimes \mathcal{O}_2 \otimes B$ contains a nonzero projection.
- (11) A is residually hereditarily in the class of all C^* -algebras B such that $\mathcal{O}_\infty \otimes B$ contains a nonzero projection.

We presume that Theorem 1.10 also holds without separability.

To put conditions (9), (10), and (11) in context, we point out that it is clear that the classes used in them are upwards directed in the sense of Definition 0.1. Applying the results of Section 5 of [29] does not give any closure properties for the collection of C^* -algebras with topological dimension zero which are not already known. We do get something new, which at least implicitly is related this characterization; see Theorem 2.6 below.

The conditions in Theorem 1.10 are not equivalent to A having the weak ideal property, since there are nonzero simple separable C^* -algebras A , such as those classified in [34], for which $K \otimes A$ has no nonzero projections. They are also not equivalent to $\text{RR}(\mathcal{O}_\infty \otimes A) = 0$. See Example 1.13 below.

Proof of Theorem 1.10. Since A has topological dimension zero if and only if $\mathcal{O}_2 \otimes A$ has topological dimension zero, and since $\mathcal{O}_2 \otimes A$ is purely infinite (by Proposition 4.5 of [16]), the equivalence of (1), (2), (3), and (4) follows by applying Theorem 1.9 to $\mathcal{O}_2 \otimes A$. Since $\mathcal{O}_\infty \otimes A$ is purely infinite (by Proposition 4.5 of [16]) and $\mathcal{O}_2 \otimes \mathcal{O}_\infty \cong \mathcal{O}_2$, the equivalence of (3), (5), and (6) follows by applying Theorem 1.9 to $\mathcal{O}_\infty \otimes A$.

We prove the equivalence of (1) and (9). Let \mathcal{C} be the class of all C^* -algebras B such that $\mathcal{O}_2 \otimes B$ contains a nonzero projection.

Assume that A has topological dimension zero; we prove that A is residually hereditarily in \mathcal{C} . Let $I \subset A$ be an ideal, and let $B \subset A/I$ be a nonzero hereditary subalgebra. Then A/I has topological dimension zero by Proposition 2.6 of [5] and Lemma 3.6 of [28]. It follows from Lemma 3.3 of [28] that B has topological dimension zero. Use (3) implies (1) in Theorem 1.9 to conclude that $\mathcal{O}_2 \otimes B$ contains a nonzero projection.

Conversely, assume that A is residually hereditarily in \mathcal{C} . We actually prove that $\mathcal{O}_2 \otimes A$ has the weak ideal property. By (5) implies (3) in Theorem 1.9, and since $\mathcal{O}_2 \otimes A$ is purely infinite (by Proposition 4.5 of [16]), it will follow that $\mathcal{O}_2 \otimes A$ has

topological dimension zero. Since $\text{Prim}(\mathcal{O}_2 \otimes A) \cong \text{Prim}(A)$, it will follow that A has topological dimension zero.

Thus, let $I \subset J \subset \mathcal{O}_2 \otimes A$ be ideals such that $J \neq I$; we must show that $K \otimes (J/I)$ contains a nonzero projection. Since \mathcal{O}_2 is simple and nuclear, there are ideals $I_0 \subset J_0 \subset A$ such that $I = \mathcal{O}_2 \otimes I_0$ and $J = \mathcal{O}_2 \otimes J_0$; moreover, $J/I \cong \mathcal{O}_2 \otimes (J_0/I_0)$. Since J_0/I_0 is a nonzero hereditary subalgebra of A/I_0 , the definition of being hereditarily in \mathcal{C} implies that $\mathcal{O}_2 \otimes (J_0/I_0)$ contains a nonzero projection, so $K \otimes (J/I) \cong K \otimes \mathcal{O}_2 \otimes (J_0/I_0)$ does also. This completes the proof of the equivalence of (1) and (9).

We prove equivalence of (9) and (11) by showing that the two classes involved are equal, that is, by showing that if B is any C^* -algebra, then $\mathcal{O}_2 \otimes B$ contains a nonzero projection if and only if $\mathcal{O}_\infty \otimes B$ contains a nonzero projection. If $\mathcal{O}_2 \otimes B$ contains a nonzero projection, use an injective (nonunital) homomorphism $\mathcal{O}_2 \rightarrow \mathcal{O}_\infty$ to produce an injective homomorphism of the minimal tensor products $\mathcal{O}_2 \otimes_{\min} B \rightarrow \mathcal{O}_\infty \otimes_{\min} B$. Since \mathcal{O}_2 and \mathcal{O}_∞ are nuclear, we have an injective homomorphism $\mathcal{O}_2 \otimes B \rightarrow \mathcal{O}_\infty \otimes B$, and hence a nonzero projection in $\mathcal{O}_\infty \otimes B$. Using an injective (unital) homomorphism from \mathcal{O}_∞ to \mathcal{O}_2 , the same argument also shows that if $\mathcal{O}_\infty \otimes B$ contains a nonzero projection then so does $\mathcal{O}_2 \otimes B$.

The proof of the equivalence of (9) and (10) is essentially the same as in the previous paragraph, using injective homomorphisms

$$\mathcal{O}_2 \longrightarrow K \otimes \mathcal{O}_2 \quad \text{and} \quad K \otimes \mathcal{O}_2 \longrightarrow \mathcal{O}_2 \otimes \mathcal{O}_2 \xrightarrow{\cong} \mathcal{O}_2.$$

We have now proved the equivalence of all the conditions except (7) and (8).

It is trivial that (6) implies (7) and that (8) implies (5).

Assume (7), so that there is a Kirchberg algebra D_0 such that $D_0 \otimes A$ has the weak ideal property. We prove (8). Let D be any Kirchberg algebra. By Theorem 1.8, the algebra $D_0 \otimes A$ has topological dimension zero. Since

$$\text{Prim}(D_0 \otimes A) \cong \text{Prim}(A) \cong \text{Prim}(D \otimes A),$$

$D \otimes A$ has topological dimension zero. Apply the already proved implication from (1) to (5) with $D \otimes A$ in place of A , concluding that $\mathcal{O}_\infty \otimes D \otimes A$ has the ideal property. Since $\mathcal{O}_\infty \otimes D \cong D$ (Theorem 3.15 of [15]), we see that $D \otimes A$ has the ideal property. \square

A naive look at condition (1) of Theorem 1.9 and the permanence properties for C^* -algebras which are residually hereditarily in some class \mathcal{C} (see Corollary 5.6 and Theorem 5.3 of [29]) might suggest that if $\mathcal{O}_\infty \otimes A$ has real rank zero and one has an arbitrary action of \mathbb{Z}_2 on $\mathcal{O}_\infty \otimes A$, or a spectrally free (Definition 1.3 of [29]) action of any discrete group on $\mathcal{O}_\infty \otimes A$, then the crossed product should also have real rank zero. This is false. We give an example of a nonsimple purely infinite unital nuclear C^* -algebra A satisfying the Universal Coefficient Theorem (in fact, with $\mathcal{O}_2 \otimes A \cong A$), with exactly one nontrivial ideal, and such that $\text{RR}(A) = 0$, and an action $\alpha: \mathbb{Z}_2 \rightarrow \text{Aut}(A)$, such that $C^*(\mathbb{Z}_2, A, \alpha)$ does not have real rank zero.

To put our example in context, we recall the following. First, Example 9 of [8] gives an example of a pointwise outer action α of \mathbb{Z}_2 on a simple unital AF algebra A such that $C^*(\mathbb{Z}_2, A, \alpha)$ does not have real rank zero. Second, by Corollary 4.4 of [12], if A is purely infinite and simple, then for any action $\alpha: \mathbb{Z}_2 \rightarrow \text{Aut}(A)$ the crossed product is again purely infinite. If α is pointwise outer, then $C^*(\mathbb{Z}_2, A, \alpha)$ is again simple, so automatically has real rank zero. Otherwise, α must be an inner action.

(See Lemma 1.11 below.) Then $C^*(\mathbb{Z}_2, A, \alpha) \cong A \oplus A$, so has real rank zero. Thus, no such example is possible when A is purely infinite and simple. Third, it is possible for A to satisfy $\mathcal{O}_2 \otimes A \cong A$ but to have $\mathcal{O}_2 \otimes C^*(\mathbb{Z}_2, A, \alpha) \not\cong C^*(\mathbb{Z}_2, A, \alpha)$. See Lemma 4.7 of [11], where this happens with $A = \mathcal{O}_2$.

The following lemma is well known, but we don't know a reference.

Lemma 1.11. Let A be a simple C^* -algebra, let G be a finite cyclic group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of G on A . Let $g_0 \in G$ be a generator of G . If α_{g_0} is inner, then α is an inner action, that is, there is a homomorphism $g \mapsto u_g$ from G to the unitary group of $M(A)$ such that $\alpha_g(a) = u_g a u_g^*$ for all $g \in G$ and $a \in A$.

Proof. Let n be the order of G .

By hypothesis, there is a unitary $v \in M(A)$ such that $\alpha_{g_0}(a) = v a v^*$ for all $a \in A$. Then $a = \alpha_{g_0}^n(a) = v^n a v^{-n}$ for all $a \in A$. Simplicity of A implies that the center of $M(A)$ contains only scalars, so there is $\lambda \in S^1$ such that $v^n = \lambda \cdot 1$. Now choose $\omega \in S^1$ such that $\omega^n = \lambda^{-1}$, giving $(\omega v)^n = 1$. Define $u_{g_0^k} = \omega^k v^k$ for $k = 0, 1, \dots, n-1$. \square

Example 1.12. There is a separable purely infinite unital nuclear C^* -algebra A with exactly one nontrivial ideal I , satisfying the Universal Coefficient Theorem, satisfying $\mathcal{O}_2 \otimes A \cong A$, and such that $\text{RR}(A) = 0$, and there is an action $\alpha: \mathbb{Z}_2 \rightarrow \text{Aut}(A)$ such that $\text{RR}(C^*(\mathbb{Z}_2, A, \alpha)) \neq 0$. Moreover, α is strongly pointwise outer in the sense of Definition 4.11 of [33] (Definition 1.1 of [29]) and spectrally free in the sense of Definition 1.3 of [29].

To start the construction, let $\nu: \mathbb{Z}_2 \rightarrow \text{Aut}(\mathcal{O}_2)$ be the action considered in Lemma 4.7 of [11]. Define $B = C^*(\mathbb{Z}_2, \mathcal{O}_2, \nu)$. Lemma 4.7 of [11] implies that B is a Kirchberg algebra (simple, separable, nuclear, and purely infinite) which is unital and satisfies the Universal Coefficient Theorem, and moreover that $K_0(B) \cong \mathbb{Z}[\frac{1}{2}]$ and $K_1(B) = 0$.

Let P be the unital Kirchberg algebra satisfying the Universal Coefficient Theorem, $K_0(P) = 0$, and $K_1(P) \cong \mathbb{Z}$. The Künneth formula (Theorem 4.1 of [36]) implies that $K_0(P \otimes \mathcal{O}_4) = 0$ and $K_1(P \otimes \mathcal{O}_4) \cong \mathbb{Z}_3$.

Since \mathcal{O}_4 satisfies the Universal Coefficient Theorem, and since $K \otimes P \otimes \mathcal{O}_4$ and \mathcal{O}_4 are separable and nuclear, every possible six term exact sequence

$$\begin{array}{ccccc} K_0(P \otimes \mathcal{O}_4) & \longrightarrow & M_0 & \longrightarrow & K_0(\mathcal{O}_4) \\ \exp \uparrow & & & & \downarrow \partial \\ K_1(\mathcal{O}_4) & \longleftarrow & M_1 & \longleftarrow & K_1(P \otimes \mathcal{O}_4) \end{array}$$

(for any possible choice of abelian groups M_0 and M_1 and homomorphisms \exp and ∂) is realized as the K-theory of an exact sequence

$$(1.1) \quad 0 \longrightarrow K \otimes P \otimes \mathcal{O}_4 \longrightarrow D \longrightarrow \mathcal{O}_4 \longrightarrow 0,$$

in which D is unital, $K_0(D) \cong M_0$, and $K_1(D) \cong M_1$. Choose the exact sequence (1.1) such that the connecting map

$$(1.2) \quad \exp: K_0(\mathcal{O}_4) \rightarrow K_1(P \otimes \mathcal{O}_4)$$

is an isomorphism. Define $A = \mathcal{O}_2 \otimes D$. Let $\iota: \mathbb{Z}_2 \rightarrow \text{Aut}(D)$ be the trivial action, and let $\alpha = \nu \otimes \iota: \mathbb{Z}_2 \rightarrow \text{Aut}(A)$ be the obvious action on the tensor product.

It follows from the isomorphisms $\mathcal{O}_2 \otimes \mathcal{O}_4 \cong \mathcal{O}_2$ and $\mathcal{O}_2 \otimes P \otimes \mathcal{O}_4 \cong \mathcal{O}_2$ that A fits into an exact sequence

$$0 \longrightarrow K \otimes \mathcal{O}_2 \longrightarrow A \longrightarrow \mathcal{O}_2 \longrightarrow 0.$$

Theorem 3.14 and Corollary 3.16 of [4] therefore imply that $\text{RR}(A) = 0$. There is also an exact sequence of crossed products

$$0 \longrightarrow C^*(\mathbb{Z}_2, \mathcal{O}_2 \otimes K \otimes P \otimes \mathcal{O}_4) \longrightarrow C^*(\mathbb{Z}_2, A, \alpha) \longrightarrow C^*(\mathbb{Z}_2, \mathcal{O}_2 \otimes \mathcal{O}_4) \longrightarrow 0,$$

in which the actions on the ideal and quotient are the tensor product of ν and the trivial action. This sequence reduces to

$$0 \longrightarrow B \otimes K \otimes P \otimes \mathcal{O}_4 \longrightarrow B \otimes D \longrightarrow B \otimes \mathcal{O}_4 \longrightarrow 0,$$

in which the maps are gotten from those of (1.1) by tensoring them with id_B . It follows from Künneth formula (Theorem 4.1 of [36]) that

$$K_0(B \otimes \mathcal{O}_4) \cong K_1(B \otimes K \otimes P \otimes \mathcal{O}_4) \cong \mathbb{Z}[\frac{1}{2}] \otimes \mathbb{Z}_3 \cong \mathbb{Z}_3.$$

By naturality, the connecting map

$$K_0(B \otimes \mathcal{O}_4) \rightarrow K_1(B \otimes K \otimes P \otimes \mathcal{O}_4)$$

is the tensor product of the isomorphism (1.2) with $\text{id}_{\mathbb{Z}[\frac{1}{2}]}$, and is hence nonzero. Since every class in $K_0(B \otimes \mathcal{O}_4)$ is represented by a projection in $B \otimes \mathcal{O}_4$, it follows from the six term exact sequence in K-theory that projections in $B \otimes \mathcal{O}_4$ need not lift to projections in $B \otimes D$. Theorem 3.14 of [4] therefore implies that $\text{RR}(B \otimes D) \neq 0$. Thus $\text{RR}(C^*(\mathbb{Z}_2, A, \alpha)) \neq 0$.

It remains to prove strong pointwise outerness and spectral freeness. In our case, these are equivalent by Theorem 1.16 of [29], so we prove strong pointwise outerness. This reduces to proving that automorphisms of $\mathcal{O}_2 \otimes K \otimes P \otimes \mathcal{O}_4$ and $\mathcal{O}_2 \otimes \mathcal{O}_4$ coming from the nontrivial element of \mathbb{Z}_2 are outer. The automorphism of \mathcal{O}_2 coming from the action ν and the nontrivial element of \mathbb{Z}_2 is outer, since otherwise the action would be inner by Lemma 1.11, so crossed product would be $\mathcal{O}_2 \oplus \mathcal{O}_2$. We can now apply Proposition 1.19 of [29] twice, both times using $\nu: \mathbb{Z}_2 \rightarrow \text{Aut}(\mathcal{O}_2)$ in place of $\alpha: G \rightarrow \text{Aut}(A)$, and in one case using $K \otimes P \otimes \mathcal{O}_4$ in place of B and in the other case using \mathcal{O}_4 .

We would like to get outerness from Theorem 1 of [37], but that theorem is only stated for unital C^* -algebras.

Example 1.13. There is a separable purely infinite unital nuclear C^* -algebra A with exactly one nontrivial ideal and which has the ideal property but such that $\mathcal{O}_\infty \otimes A$ does not have real rank zero.

Let D be as in (1.1) in Example 1.12, with the property that the connecting map in (1.2) is nonzero. Set $A = \mathcal{O}_\infty \otimes D$. Since $\mathcal{O}_\infty \otimes K \otimes P \otimes \mathcal{O}_4$ and $\mathcal{O}_\infty \otimes \mathcal{O}_4$ have the weak ideal property (for trivial reasons), it follows from Theorem 8.5(5) of [29] that A has the weak ideal property, and from Theorem 1.9 that A has the ideal property. However, A is by construction not K_0 -liftable in the sense of Definition 3.1 of [31], so Corollary 4.3 of [31] implies that $\mathcal{O}_\infty \otimes A$ (which is of course isomorphic to A) does not have real rank zero.

2. PERMANENCE PROPERTIES FOR CROSSED PRODUCTS

In [29] we proved that if \mathcal{C} is an upwards directed class of C^* -algebras, α is a completely arbitrary action of \mathbb{Z}_2 on a C^* -algebra A , and A^α is (residually) hereditarily in \mathcal{C} , then A is (residually) hereditarily in \mathcal{C} . (See Theorem 5.5 of [29].) In particular, by considering dual actions, it follows (Corollary 5.6 of [29]) that crossed products by arbitrary actions of \mathbb{Z}_2 preserve the class of C^* -algebras which are (residually) hereditarily in \mathcal{C} . Here, we show how one can easily extend the first result to arbitrary groups of order a power of 2 and the second result to arbitrary abelian groups of order a power of 2. This should have been done in [29], but was overlooked there. We believe these results should be true for any finite group in place of \mathbb{Z}_2 , or at least any finite abelian group, but we don't know how to prove them in this generality.

The following lemma is surely well known.

Lemma 2.1. Let G be a topological group, let A be a C^* -algebra, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of G on A . Let $N \subset G$ be a closed normal subgroup. Then there is an action $\bar{\alpha}: G/N \rightarrow \text{Aut}(A^{\alpha|_N})$ such that for $g \in G$ and $a \in A^{\alpha|_N}$ we have $\bar{\alpha}_{gN}(a) = \alpha_g(a)$. Moreover, $(A^{\alpha|_N})^{\bar{\alpha}} = A^\alpha$.

Proof. The only thing requiring proof is that if $g \in G$ and $a \in A^{\alpha|_N}$ then $\alpha_g(a) \in A^{\alpha|_N}$. So let $k \in N$. Since $g^{-1}kg \in N$, we get

$$\alpha_k(\alpha_g(a)) = \alpha_g(\alpha_{g^{-1}kg}(a)) = \alpha_g(a).$$

This completes the proof. \square

Theorem 2.2. Let \mathcal{C} be an upwards directed class of C^* -algebras. Let G be a finite 2-group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an arbitrary action of G on a C^* -algebra A .

- (1) If A^α is hereditarily in \mathcal{C} , then A is hereditarily in \mathcal{C} .
- (2) If A^α is residually hereditarily in \mathcal{C} , then A is residually hereditarily in \mathcal{C} .

Proof. We prove both parts at once.

We use induction on the number $n \in \mathbb{Z}_{\geq 0}$ such that the order of G is 2^n . When $n = 0$, the statement is trivial. So assume $n \in \mathbb{Z}_{\geq 0}$, the statement is known for all groups of order 2^n , G is a group with $\text{card}(G) = 2^{n+1}$, A is a C^* -algebra, $\alpha: G \rightarrow \text{Aut}(A)$ is an action, and A^α is (residually) hereditarily in \mathcal{C} . The Sylow Theorems provide a subgroup $N \subset G$ such that $\text{card}(N) = 2^n$. Since N has index 2, N must be normal. Let $\bar{\alpha}: G/N \rightarrow \text{Aut}(A^{\alpha|_N})$ be as in Lemma 2.1. Then $(A^{\alpha|_N})^{\bar{\alpha}} = A^\alpha$ is (residually) hereditarily in \mathcal{C} . Since $G/N \cong \mathbb{Z}_2$, it follows from Theorem 5.5 of [29] that $A^{\alpha|_N}$ is (residually) hereditarily in \mathcal{C} . The induction hypothesis now implies that A is (residually) hereditarily in \mathcal{C} . \square

Corollary 2.3. Let \mathcal{C} be an upwards directed class of C^* -algebras. Let G be a finite abelian 2-group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an arbitrary action of G on a C^* -algebra A .

- (1) If A is hereditarily in \mathcal{C} , then $C^*(G, A, \alpha)$ and A^α are hereditarily in \mathcal{C} .
- (2) If A is residually hereditarily in \mathcal{C} , then $C^*(G, A, \alpha)$ and A^α are residually hereditarily in \mathcal{C} .

Proof. For $C^*(G, A, \alpha)$, apply Theorem 2.2 with $C^*(G, A, \alpha)$ in place of A and the dual action $\hat{\alpha}$ in place of α .

For A^α , use the Proposition in [35] to see that A^α is isomorphic to a corner of $C^*(G, A, \alpha)$, and apply Proposition 5.10 of [29]. \square

Presumably Corollary 2.3 is valid for crossed products by coactions of not necessarily abelian 2-groups. Indeed, possibly the appropriate context is that of actions of finite dimensional Hopf C^* -algebras. We will not pursue this direction here.

Corollary 2.4. Let G be a finite 2-group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an arbitrary action of G on a C^* -algebra A . Suppose A^α has one of the following properties: residual hereditary infiniteness, residual hereditary proper infiniteness, residual (SP), or the combination of the ideal property and pure infiniteness. Then A has the same property.

Proof. As discussed in the introduction, for each of these properties there is an upwards directed class \mathcal{C} such that a C^* -algebra has the property if and only if it is residually hereditarily in the class \mathcal{C} . Apply Theorem 2.2. \square

Corollary 2.5. Let G be a finite abelian 2-group, and let $\alpha: G \rightarrow \text{Aut}(A)$ be an arbitrary action of G on a C^* -algebra A . Suppose A has one of the following properties: residual hereditary infiniteness, residual hereditary proper infiniteness, residual (SP), or the combination of the ideal property and pure infiniteness. Then $C^*(G, A, \alpha)$ and A^α have the same property.

Proof. The proof is the same as that of Corollary 2.4, using Corollary 2.3 instead of Theorem 2.2. \square

We omit the weak ideal property in Corollary 2.4 and Corollary 2.5, because better results are already known (Theorem 8.9 and Corollary 8.10 of [29]). We also already know (Theorem 3.17 of [28]) that topological dimension zero is preserved by crossed products by actions of arbitrary finite abelian groups, not just abelian 2-groups. The result analogous to Corollary 2.4 is Theorem 3.14 of [28], but it has an extra technical hypothesis. In the separable case, we remove this hypothesis.

Theorem 2.6. Let $\alpha: G \rightarrow \text{Aut}(A)$ be an action of a finite group G on a separable C^* -algebra A . Suppose that A^α has topological dimension zero. Then A has topological dimension zero.

Proof. Define an action $\beta: G \rightarrow \text{Aut}(\mathcal{O}_2 \otimes A)$ by $\beta_g = \text{id}_{\mathcal{O}_2} \otimes \alpha_g$ for $g \in G$. The implication from (1) to (4) in Theorem 1.10 shows that $(\mathcal{O}_2 \otimes A)^\beta = \mathcal{O}_2 \otimes A^\alpha$ has the weak ideal property. Theorem 8.9 of [29] now implies that $\mathcal{O}_2 \otimes A$ has the weak ideal property. So A has topological dimension zero by the implication from (4) to (1) in Theorem 1.10. \square

3. PERMANENCE PROPERTIES FOR TENSOR PRODUCTS

In this section, we consider permanence properties for tensor products. One of its purposes is to serve as motivation for the results on $C_0(X)$ -algebras in Section 4. The new positive result is Theorem 3.6: if A and B are separable C^* -algebras and A is exact, then $A \otimes_{\min} B$ has topological dimension zero if and only if A and B have topological dimension zero. The exactness hypothesis is necessary (Example 3.3). Still assuming this exactness hypothesis, we also give partial results for the weak ideal property, when one of the tensor factors actually has the ideal property and both are separable (Theorem 3.10), and when one of them has finite or Hausdorff primitive ideal space (Proposition 3.12 and Proposition 3.13).

The following two easy examples show that the properties we are considering are certainly not preserved by taking tensor products with arbitrary C^* -algebras.

Example 3.1. The algebra \mathbb{C} has all of topological dimension zero, the ideal property, the weak ideal property, and residual (SP), but $C([0, 1]) \otimes \mathbb{C}$ has none of these.

Example 3.2. The algebra \mathcal{O}_2 is purely infinite and has the ideal property but $C([0, 1]) \otimes \mathcal{O}_2$ does not have the ideal property.

In particular, there is no hope of any general theorem about tensor products for properties of the form “residually hereditarily in \mathcal{C} ” when only one tensor factor has the property. Permanence theorems will therefore have to assume that both factors have the property in question. The following example shows that we will also need to assume that at least one tensor factor is exact.

Example 3.3. We show that there are separable unital C^* -algebras A and C (neither of which is exact) which have topological dimension zero and such that $A \otimes_{\min} C$ does not have topological dimension zero. In fact, A and C even have real rank zero, and C is simple. We also show that there are separable unital C^* -algebras B and D which are purely infinite and have the ideal property, but such that $B \otimes_{\min} D$ does not have the ideal property. In fact, B and D even tensorially absorb \mathcal{O}_2 , and D is simple.

Since topological dimension zero and the weak ideal property are preserved by passing to quotients, it follows that no other tensor product of A and C has topological dimension zero, and that no other tensor product of B and D even has the weak ideal property.

Let A and C be as in Theorem 2.6 of [30]. As there, A and C are separable unital C^* -algebras with real rank zero, C is simple, and $A \otimes_{\min} C$ does not have the ideal property. These are the same algebras A and C as used in the proof of Proposition 4.5 of [31]. Thus, $A \otimes_{\min} C$ does not have topological dimension zero by Proposition 4.5(1) of [31]. Also, $\mathcal{O}_2 \otimes A \otimes_{\min} C$ does not have the ideal property by Proposition 4.5(2) of [31]. Thus taking $B = \mathcal{O}_2 \otimes A$ and $D = \mathcal{O}_2 \otimes C$ gives algebras B and D with the required properties.

We have several positive results, but no answers for several obvious questions. We recall known results, then give the new result we can prove (on topological dimension zero) and our partial results for the weak ideal property. We conclude with open questions.

In order to get

$$(3.1) \quad \text{Prim}(A \otimes_{\min} B) \cong \text{Prim}(A) \times \text{Prim}(B).$$

we will assume one of the algebras is exact and both are separable. In Theorem 3.6, Theorem 3.10, and Corollary 3.11, these assumptions can be replaced by any other hypotheses which imply a natural homeomorphism as in (3.1). Proposition 2.17 of [2] gives a number of conditions which imply this for the spaces of prime ideals in place of the primitive ideal spaces, but for separable C^* -algebras this is the same thing.

Theorem 3.4 (Corollary 1.3 of [30]). Let A and B be C^* -algebras with the ideal property. Assume that A is exact. Then $A \otimes_{\min} B$ has the ideal property.

Theorem 3.5 (Proposition 4.6 of [31]). Let A and B be C^* -algebras with the ideal property. Assume that B is purely infinite and A is exact. Then $A \otimes_{\min} B$ is purely infinite and has the ideal property.

Theorem 3.6. Let A and B be separable C^* -algebras. Assume that A is exact. Then $A \otimes_{\min} B$ has topological dimension zero if and only if both A and B have topological dimension zero.

Proof. By Proposition 2.17 of [2] (see Remark 2.11 of [2] for the notation in Proposition 2.16 of [2], to which it refers), the spaces of closed prime ideals satisfy

$$\text{prime}(A \otimes_{\min} B) \cong \text{prime}(A) \times \text{prime}(B),$$

with the homeomorphism being implemented in the obvious way. (See Proposition 2.16(iii) of [2].) Since A , B , and $A \otimes_{\min} B$ are all separable, Proposition 4.3.6 of [32] implies that prime ideals are primitive; the reverse is well known. So

$$(3.2) \quad \text{Prim}(A \otimes_{\min} B) \cong \text{Prim}(A) \times \text{Prim}(B).$$

Assume A and B have topological dimension zero. Then (see Definition 1.1) we need to prove that if X and Y are locally compact but not necessarily Hausdorff spaces which have topological dimension zero, then $X \times Y$ has topological dimension zero. So let $(x, y) \in X \times Y$, and let $W \subset X \times Y$ be an open set with $(x, y) \in W$. By the definition of the product topology, there are open subsets $U_0 \subset X$ and $V_0 \subset Y$ such that $x \in U_0$ and $y \in V_0$. By the definition of topological dimension zero, there are compact open (but not necessarily closed) subsets $U \subset U_0$ and $V \subset V_0$ such that $x \in U$ and $y \in V$. Then $U \times V$ is a compact open subset of $X \times Y$ such that $(x, y) \in U \times V \subset W$.

Now assume $A \otimes_{\min} B$ has topological dimension zero. We prove that B has topological dimension zero; the proof that A has topological dimension zero is the same, except that we don't need to know that exactness passes to quotients. Choose a maximal ideal $I \subset A$. Then A/I is also exact, by Proposition 7.1(ii) of [13]. Apply (3.2) as is and also with A/I in place of A , use the formula for the homeomorphism from Proposition 2.16(iii) of [2], and use the quotient map $A \rightarrow A/I$ and the map $\text{Prim}((A/I) \otimes_{\min} B) \rightarrow \text{Prim}(A \otimes_{\min} B)$ it induces. The outcome is that

$$\begin{aligned} \text{Prim}(B) &\cong \text{Prim}((A/I) \otimes_{\min} B) \cong \{I\} \times \text{Prim}(B) \\ &\subset \text{Prim}(A) \times \text{Prim}(B) \cong \text{Prim}(A \otimes_{\min} B), \end{aligned}$$

and is a closed subset. Combining Lemmas 3.6 and 3.8 of [28], we see that closed subsets of spaces with topological dimension zero also have topological dimension zero. \square

The first result for the weak ideal property requires some preparation.

Notation 3.7. Let A be a C^* -algebra. For an open set $U \subset \text{Prim}(A)$, we let $I_A(U) \subset A$ be the corresponding ideal. Thus

$$\text{Prim}(I_A(U)) \cong U \quad \text{and} \quad \text{Prim}(A/I_A(U)) \cong \text{Prim}(A) \setminus U.$$

Lemma 3.8. Let A be a C^* -algebra, let $U \subset \text{Prim}(A)$ be open, and let $p \in A/I_A(U)$ be a projection. Then there exist an open subset $V \subset \text{Prim}(A)$, a compact (but not necessarily closed) subset $L \subset \text{Prim}(A)$, and a projection $q \in A/I_A(V)$, such that $V \subset L \subset U$ and the image of q in $A/I_A(U)$ is equal to p .

Proof. For $P \in \text{Prim}(A)$ let $\pi_P: A \rightarrow A/P$ be the quotient map, and for an open subset $W \subset \text{Prim}(A)$ let $\kappa_W: A \rightarrow A/I_A(W)$ be the quotient map. Choose $a \in A_{\text{sa}}$ such that $\kappa_U(a) = p$. Define

$$V = \{P \in \text{Prim}(A): \|\pi_P(a^2 - a)\| > \frac{1}{8}\}$$

and

$$L = \{P \in \text{Prim}(A): \|\pi_P(a^2 - a)\| \geq \frac{1}{8}\}.$$

We apply results in [7] to these sets. These results are actually stated in terms of functions on the space \widehat{A} of unitary equivalence classes of irreducible representations of A , with the topology being the inverse image of the topology on $\text{Prim}(A)$ under the standard surjection $\widehat{A} \rightarrow \text{Prim}(A)$, but they clearly apply to $\text{Prim}(A)$. It follows from Proposition 3.3.2 of [7] that V is open, and from Proposition 3.3.7 of [7] that L is compact. Obviously $V \subset L$. Clearly $\pi_P(a^2 - a) = 0$ for all $P \in \text{Prim}(A) \setminus U$, so $L \subset U$.

Lemma 3.3.6 of [7] implies that $\|\kappa_V(a^2 - a)\| \leq \frac{1}{8}$. Therefore $\frac{1}{2} \notin \text{sp}(\kappa_V(a))$. Thus we can define a projection $q \in A/I_A(V)$ by $q = \chi_{(\frac{1}{2}, \infty)}(\kappa_V(a))$. The image of q in $A/I_A(U)$ is clearly equal to p . \square

Lemma 3.9. Let X_1 and X_2 be topological spaces, let $W \subset X_1 \times X_2$ be an open subset, let $x \in X_1$, let $L \subset X_2$ be compact, and suppose that $\{x\} \times L \subset W$. Then there exists an open set $U \subset X_1$ such that $x \in U$ and $U \times L \subset W$.

We don't assume that X_1 and X_2 are Hausdorff. In particular, L need not be closed.

Proof of Lemma 3.9. For each $y \in L$, choose open sets $V_1(y) \subset X_1$ and $V_2(y) \subset X_2$ such that $(x, y) \in V_1(y) \times V_2(y) \subset W$. Use compactness of L to choose $n \in \mathbb{Z}_{\geq 0}$ and $y_1, y_2, \dots, y_n \in L$ such that $V_2(y_1), V_2(y_2), \dots, V_2(y_n)$ cover L . Take $U = \bigcap_{j=1}^n V_1(y_j)$. \square

Theorem 3.10. Let A_1 and A_2 be separable C^* -algebras. Assume that A_1 or A_2 is exact, that A_1 has the ideal property, and that A_2 has the weak ideal property. Then $A_1 \otimes_{\min} A_2$ has the weak ideal property.

In the diagram (3.5) in the proof below, one should think of the subquotients as corresponding to locally closed subsets of $\text{Prim}(A_1) \times \text{Prim}(A_2)$. Thus, the algebra in the middle of the top row corresponds to $V_1 \times (V_2 \setminus T)$. It contains a useful nonzero projection, obtained as the tensor product of suitable projections in $I_{A_1}(V_1)$ and $I_{A_2}(V_2)/I_{A_2}(T)$. This subset isn't open, so the algebra isn't a subalgebra of $A_1 \otimes_{\min} A_2$. A main point in the proof is that, given V_2 and a nonzero projection $e_2 \in I_{A_2}(V_2)/I_{A_2}(S_2)$ (see (3.3) for the definition of S_2), the sets V_1 and T have been chosen so that there is a projection $p_2 \in I_{A_2}(V_2)/I_{A_2}(T)$ whose image is e_2 , and so that the set $Y \cup [V_1 \times (V_2 \setminus T)]$ is open.

We do not get a proof the the tensor product of two algebras with the weak ideal property again has the weak ideal property, because we do not know how to reduce the size of V_2 (to go with an analogous subset $T_1 \subset V_1$) without changing the projection e_2 .

Proof of Theorem 3.10. Replacing A_2 by $K \otimes A_2$, we may assume that whenever $I, J \subset A_2$ are ideals such that $I \subsetneq J$, then J/I contains a nonzero projection.

Define $X_j = \text{Prim}(A_j)$ for $j = 1, 2$. Using [2] in the same way as in the proof of Theorem 3.6, we identify

$$\text{Prim}(A_1 \otimes_{\min} A_2) = X_1 \times X_2.$$

The identification is given by the map from $X_1 \times X_2$ to $\text{Prim}(A_1 \otimes_{\min} A_2)$ sending $(P_1, P_2) \in X_1 \times X_2$ to the primitive ideal obtained as the kernel of $A_1 \otimes_{\min} A_2 \rightarrow (A_1/P_1) \otimes_{\min} (A_2/P_2)$. The lattice of ideals of $A_1 \otimes_{\min} A_2$ can thus be canonically identified with the lattice of open subsets of $X_1 \times X_2$, when this space is equipped with the product topology. We simplify Notation 3.7 by writing $I_j(U)$ for $I_{A_j}(U)$ when $U \subset X_j$ is open, and $I(W)$ for $I_{A_1 \otimes_{\min} A_2}(W)$ when $W \subset X_1 \times X_2$ is open. We then get canonical isomorphisms $I_1(U_1) \otimes_{\min} I_2(U_2) \cong I(U_1 \times U_2)$ for open subsets $U_1 \subset X_1$ and $U_2 \subset X_2$.

We need to show that if $Y, Z \subset X_1 \times X_2$ are open subsets such that $Y \subsetneq Z$, then $I(Z)/I(Y)$ contains a nonzero projection.

Choose $x_1 \in X_1$ and $x_2 \in X_2$ such that $(x_1, x_2) \in Z \setminus Y$. Choose open sets $U \subset X_1$ and $V_2 \subset X_2$ such that

$$x_1 \in U, \quad x_2 \in V_2, \quad \text{and} \quad U \times V_2 \subset Z.$$

Define

$$(3.3) \quad S_2 = \{y \in V_2 : (x_1, y) \in Y\},$$

which is an open subset of V_2 . Since A_2 has the weak ideal property, there is a nonzero projection $e_2 \in I_2(V_2)/I_2(S_2)$. Use Lemma 3.8 to choose subsets $T \subset L \subset S_2 \subset V_2$ such that L is compact, T is open, and there is a projection $p_2 \in I_2(V_2)/I_2(T)$ whose image in $I_2(V_2)/I_2(S_2)$ is equal to e_2 . Use Lemma 3.9 to choose an open set $V_1 \subset U$ such that $x_1 \in V_1$ and $V_1 \times L \subset Y$. Define $S_1 = V_1 \cap (X_1 \setminus \overline{\{x_1\}})$, which is an open subset of V_1 . Since A_1 has the ideal property, there is a projection $p_1 \in I_1(V_1)$ whose image $e_1 \in I_1(V_1)/I_1(S_1)$ is nonzero.

We claim that

$$(3.4) \quad Y \cap [(V_1 \setminus S_1) \times (V_2 \setminus S_2)] = \emptyset.$$

The definitions of the sets involved imply that

$$V_1 \setminus S_1 \subset \overline{\{x_1\}} \quad \text{and} \quad V_2 \setminus S_2 = \{y \in V_2 : (x_1, y) \notin Y\}.$$

Therefore

$$(V_1 \setminus S_1) \times (V_2 \setminus S_2) \subset \overline{\{x_1\} \times (V_2 \setminus S_2)} \quad \text{and} \quad [\{x_1\} \times (V_2 \setminus S_2)] \cap Y = \emptyset.$$

Since Y is open, the claim follows.

We now want to construct a commutative diagram as follows:

$$(3.5) \quad \begin{array}{ccccc} I(V_1 \times V_2) & \xrightarrow{\pi} & I(V_1 \times V_2)/I(V_1 \times T) & \xrightarrow{\kappa} & I(V_1 \times V_2)/I(R) \\ \downarrow \iota & & \downarrow \varphi & & \downarrow \psi \\ I(Z) & \xrightarrow{\sigma} & I(Z)/I(Y) & \xrightarrow{\rho} & I(Z)/I(Y \cup R). \end{array}$$

The maps π and σ are the obvious quotient maps, and ι is the obvious inclusion, coming from $V_1 \times V_2 \subset U \times V_2 \subset Z$. We define $R = (V_1 \times S_2) \cup (S_1 \times V_2)$, which is an open subset of $V_1 \times V_2$. In particular, $R \subset Z$. The map κ is then the quotient map arising from the inclusion $V_1 \times T \subset V_1 \times S_2 \subset R$, and ρ is the quotient map arising from the inclusion $Y \subset Y \cup R$. Since π is surjective, the map φ is unique

if it exists. For existence, we must show that $\text{Ker}(\pi) \subset \text{Ker}(\sigma \circ \iota)$. This inclusion follows from

$$\text{Ker}(\pi) = I(V_1 \times T), \quad \text{Ker}(\sigma \circ \iota) = I((V_1 \times V_2) \cap Y), \quad \text{and} \quad V_1 \times T \subset V_1 \times L \subset Y.$$

It remains to construct ψ . Since κ is surjective, the map ψ is unique if it exists. We claim that $\text{Ker}(\kappa) = \text{Ker}(\rho \circ \varphi)$. Since π is surjective, it suffices to prove that $\text{Ker}(\kappa \circ \pi) = \text{Ker}(\rho \circ \varphi \circ \pi)$. We easily check that

$$\text{Ker}(\kappa \circ \pi) = I(R) \quad \text{and} \quad \text{Ker}(\rho \circ \varphi \circ \pi) = I((Y \cup R) \cap (V_1 \times V_2)).$$

It follows from (3.4) that $(Y \cup R) \cap (V_1 \times V_2) = R$, proving the claim. The claim implies not only that there is a map ψ making the right hand square commute, but also that ψ is injective.

The identification $\text{Prim}(A_1 \otimes_{\min} A_2) = X_1 \times X_2$ gives identifications

$$I(V_1 \times V_2)/I(V_1 \times T) = I(V_1) \otimes_{\min} [I(V_2)/I(T)]$$

and

$$I(V_1 \times V_2)/I(R) = [I(V_1)/I(S_1)] \otimes_{\min} [I(V_2)/I(S_2)],$$

with respect to which κ becomes the tensor product of the quotient maps

$$I(V_1) \rightarrow I(V_1)/I(S_1) \quad \text{and} \quad I(V_2)/I(T) \rightarrow I(V_2)/I(S_2).$$

Define $q \in I(Z)/I(Y)$ by $q = \varphi(p_1 \otimes p_2)$. Then q is a projection. Moreover,

$$\rho(q) = (\psi \circ \kappa)(p_1 \otimes p_2) = \psi(e_1 \otimes e_2).$$

Since $e_1 \neq 0$, $e_2 \neq 0$, and ψ is injective, it follows that $q \neq 0$. Thus $I(Z)/I(Y)$ contains a nonzero projection, as desired. \square

Using results from Section 6 below, we can now give a case in which the tensor product of C*-algebras with the weak ideal property again has this property.

Corollary 3.11. Let A and B be separable C*-algebras. Assume that A or B is exact, and that A is in the class \mathcal{W} of Theorem 6.12. If A and B have the weak ideal property, then $A \otimes_{\min} B$ has the weak ideal property.

The class \mathcal{W} is the smallest class of separable C*-algebras which contains the separable locally AH algebras, the separable LS algebras, the separable type I C*-algebras, and the separable purely infinite C*-algebras, and is closed under finite and countable direct sums and under minimal tensor products when one tensor factor is exact.

Proof of Corollary 3.11. By Theorem 1.8, the algebra A has topological dimension zero. Combine Lemma 6.5, Lemma 6.6, Lemma 6.10, Lemma 6.9(2), Proposition 6.11, and Theorem 1.9, to see that A is in the class \mathcal{P} of Notation 6.3. Thus A has the ideal property. So $A \otimes_{\min} B$ has the weak ideal property by Theorem 3.10. \square

Combining Proposition 6.13 below with Theorem 3.10 and with Theorem 8.5(6) of [29], one immediately sees that if A and B are separable C*-algebras with the weak ideal property, one of which is exact, and $\text{Prim}(A)$ is Hausdorff, then $A \otimes_{\min} B$ has the weak ideal property. A different argument allows one to prove this without separability. We give it here, although it is based on material on $C_0(X)$ -algebras in the next section. We first consider the case in which $\text{Prim}(A)$ is finite.

Proposition 3.12. Let A and B be C^* -algebras with the weak ideal property such that $\text{Prim}(A)$ is finite and A or B is exact. Then $A \otimes_{\min} B$ has the weak ideal property.

Proof. First suppose that A is simple. Using Proposition 2.17(2) and parts (ii) and (iv) of Proposition 2.16 of [2], we see that $J \mapsto A \otimes_{\min} J$ is a one to one correspondence from the ideals of B to the ideals of $A \otimes_{\min} B$; moreover, if $J_1 \subset J_2 \subset B$ are ideals, then

$$(A \otimes_{\min} J_2)/(A \otimes_{\min} J_1) \cong A \otimes_{\min} (J_2/J_1).$$

Now let $L_1, L_2 \subset A \otimes_{\min} B$ be ideals with $L_1 \subset L_2$. It follows that there exist ideals $J_1, J_2 \subset B$ with $J_1 \subset J_2$ such that $L_2/L_1 \cong A \otimes_{\min} (J_2/J_1)$. There are nonzero projections $p_1 \in K \otimes A$ and $p_2 \in K \otimes (J_2/J_1)$, so $p_1 \otimes p_2$ is a nonzero projection in

$$[K \otimes A] \otimes_{\min} [K \otimes (J_2/J_1)] \cong K \otimes (L_2/L_1).$$

We prove the general case by induction on $\text{card}(\text{Prim}(A))$. We just did the case $\text{card}(\text{Prim}(A)) = 1$. So let $n \in \mathbb{Z}_{>0}$ and suppose the result is known whenever $\text{card}(\text{Prim}(A)) < n$. Suppose that $\text{card}(\text{Prim}(A)) = n$. Choose a nontrivial ideal $I \subset A$. By Proposition 2.17(2) and Proposition 2.16(iv) of [2], the sequence

$$0 \longrightarrow I \otimes_{\min} B \longrightarrow A \otimes_{\min} B \longrightarrow (A/I) \otimes_{\min} B \longrightarrow 0$$

is exact. The algebras $I \otimes_{\min} B$ and $(A/I) \otimes_{\min} B$ have the weak ideal property by the induction hypothesis, so $A \otimes_{\min} B$ has the weak ideal property by Theorem 8.5(5) of [29]. \square

Much of the proof of the following proposition will be reused in the proof of Proposition 6.13 below.

Proposition 3.13. Let A and B be C^* -algebras such that A or B is exact, and $\text{Prim}(A)$ is Hausdorff. If A and B have the weak ideal property, then $A \otimes_{\min} B$ has the weak ideal property.

Proof. Set $X = \text{Prim}(A)$. We first claim that A is a continuous $C_0(X)$ -algebra with fiber $A_P = A/P$ for $P \in X$. In the language of continuous fields, this is Theorem 2.3 of [9]. To get it in our language, apply Theorem 3.3 of [20], taking $\alpha: \text{Prim}(A) \rightarrow X$ to be the identity map. Identifying continuous $C_0(X)$ -algebras and continuous C^* -bundles as in Proposition 4.6, we use Corollary 2.8 of [18] to see that $A \otimes_{\min} B$ is a continuous $C_0(X)$ -algebra, with fibers $(A \otimes_{\min} B)_P = (A/P) \otimes_{\min} B$ for $P \in X$.

The algebra A has topological dimension zero by Theorem 1.8. Since X is Hausdorff, it follows that X is totally disconnected. For every $P \in X$, the quotient A/P is simple because $\{P\}$ is closed, and has the weak ideal property by Theorem 8.5(5) of [29]. So the fiber $(A \otimes_{\min} B)_P = (A/P) \otimes_{\min} B$ has the weak ideal property by Proposition 3.12. Theorem 4.14(3) now implies that $A \otimes_{\min} B$ has the weak ideal property. \square

Question 3.14. Let A and B be C^* -algebras, with A exact. If A and B have the weak ideal property, does $A \otimes_{\min} B$ have the weak ideal property?

Question 3.15. Let A and B be C^* -algebras, with A exact. If A and B have residual (SP), does $A \otimes_{\min} B$ have residual (SP)?

4. PERMANENCE PROPERTIES FOR BUNDLES OVER TOTALLY DISCONNECTED SPACES

We now turn to section algebras of continuous fields over totally disconnected base spaces. We prove that if A is the section algebra of a bundle over a totally disconnected space, and the fibers all have one of the properties residual (SP), topological dimension zero, the weak ideal property, or the combination of the ideal property and pure infiniteness, then A also has the same property. Moreover, if A has one of these properties, so do all the fibers.

The section algebra of a continuous field over a space which is not totally disconnected will not have the weak ideal property except in trivial cases, and the same is true of the other properties involving the existence of projections in ideals. See Example 3.1 and Example 3.2, showing that this fails even for trivial continuous fields. Accordingly, we can't drop the requirement that the base space be totally disconnected. Indeed, we prove that for a continuous field over a second countable locally compact Hausdorff space with nonzero fibers, if the section algebra is separable and has one of the four properties above then the base space must be totally disconnected.

The fact that the properties we consider are equivalent to being residually hereditarily in a suitable class \mathcal{C} underlies some of our reasoning, but knowing that a property has this form does not seem to be sufficient for our results.

Following standard notation, if A is a C^* -algebra then $M(A)$ is its multiplier algebra and $Z(A)$ is its center.

Definition 4.1. Let X be a locally compact Hausdorff space. Then a $C_0(X)$ -algebra is a C^* -algebra A together with a nondegenerate (see below) homomorphism $\iota: C_0(X) \rightarrow Z(M(A))$. Here ι is *nondegenerate* if $\overline{\iota(C_0(X))A} = A$.

Unlike in Definition 2.1 of [20], we do not assume that ι is injective. This permits a hereditary subalgebra of A to also be a $C_0(X)$ -algebra, without having to replace X by a closed subspace.

Notation 4.2. Let the notation be as in Definition 4.1. For an open set $U \subset X$ we identify $C_0(U)$ with the obvious ideal of $C_0(X)$. Then $\overline{\iota(C_0(U))A}$ is an ideal in A . For $x \in X$, we define

$$A_x = A / \overline{\iota(C_0(X \setminus \{x\}))A},$$

and we let $\text{ev}_x: A \rightarrow A_x$ be the quotient map. For a closed subset $L \subset X$, we define $A|_L = A / \overline{\iota(C_0(X \setminus L))A}$. We equip it with the $C_0(L)$ -algebra structure which comes from the fact that $C_0(X \setminus L)$ is contained in the kernel of the composition

$$C_0(X) \longrightarrow Z(M(A)) \longrightarrow Z(M(A|_L)).$$

Thus $A_x = A|_{\{x\}}$. Strictly speaking, A is the section algebra of a bundle and $A|_L$ is the section algebra of the restriction of this bundle to L , but the abuse of notation is convenient.

Lemma 4.3. Let the notation be as in Definition 4.1 and Notation 4.2. Let $a \in A$. Then:

- (1) $\|a\| = \sup_{x \in X} \|\text{ev}_x(a)\|$.
- (2) For every $\varepsilon > 0$, the set $\{x \in X: \|\text{ev}_x(a)\| \geq \varepsilon\} \subset X$ is compact.
- (3) The function $x \mapsto \|\text{ev}_x(a)\|$ is upper semicontinuous.

- (4) For $f \in C_0(X)$ and $x \in X$, we have $\text{ev}_x(\iota(f)a) = f(x)\text{ev}_x(a)$.

Proof. When ι is injective, the first three parts are Corollary 2.2 of [20], and the last part is contained in the proof of Theorem 2.3 of [20]. In the general case, let $Y \subset X$ be the closed subset such that

$$\text{Ker}(\iota) = \{f \in C_0(X) : f|_Y = 0\}.$$

Then A is a $C_0(Y)$ -algebra in the obvious way. We have $A_x = 0$ for $x \notin Y$, and the function $x \mapsto \|\text{ev}_x(a)\|$ associated with the $C_0(X)$ -algebra structure is gotten by extending the one associated with the $C_0(Y)$ -algebra structure to be zero on $X \setminus Y$. The first three parts then follow from those for the $C_0(Y)$ -algebra structure, as does the last when $x \in Y$. The last part is trivial for $x \in X \setminus Y$. \square

Definition 4.4. Let X be a locally compact Hausdorff space, and let A be a $C_0(X)$ -algebra. We say that A is a *continuous $C_0(X)$ -algebra* if for all $a \in A$, the map $x \mapsto \|\text{ev}_x(a)\|$ of Lemma 4.3(3) is continuous.

Proposition 4.5. Let X be a locally compact Hausdorff space and let A be a C^* -algebra. Then homomorphisms $\iota : C_0(X) \rightarrow Z(M(A))$ which make A is a continuous $C_0(X)$ -algebra correspond bijectively to isomorphisms of A with the algebra of continuous sections vanishing at infinity of a continuous field of C^* -algebras over X , as in 10.4.1 of [7].

Proof. This is essentially contained in Theorem 2.3 of [20], referring to the definitions at the end of Section 1 of [20]. \square

We will also need to use results from [18], so we compare definitions.

Proposition 4.6. Let X be a locally compact Hausdorff space.

- (1) Let $(X, (\pi_x : A \rightarrow A_x)_{x \in X}, A)$ be a C^* -bundle in the sense of Definition 1.1 of [18]. Then A is a $C_0(X)$ -algebra, with structure map $\iota : C_0(X) \rightarrow Z(M(A))$ determined by the product in Definition 1.1(ii) of [18], if and only if for every $a \in A$ the function $x \mapsto \|\pi_x(a)\|$ is upper semicontinuous and vanishes at infinity.
- (2) Let A be a $C_0(X)$ -algebra. Then $(X, (\text{ev}_x : A \rightarrow A_x)_{x \in X}, A)$ is a C^* -bundle in the sense of Definition 1.1 of [18] which satisfies the condition in (1).
- (3) In (1) and (2), A is a continuous $C_0(X)$ -algebra if and only if the corresponding C^* -bundle is continuous in the sense of Definition 1.1(iii) of [18].

Proof. Theorem 2.3 of [20] and the preceding discussion gives a one to one correspondence between $C_0(X)$ -algebras and upper semicontinuous bundles over X in the sense of the definitions at the end of Section 1 of [20]. (The version stated there is for a special case: the structure map of the $C_0(X)$ -algebra is required to be injective and the fibers of the bundle are required to be nonzero on a dense subset of X . But the argument in [20] also proves the general case. Some of the argument is also contained in Lemma 2.1 of [18].)

The difference between Definition 1.1 of [18] and the definition of [20] is that [18] omits the requirement (condition (ii) in [20]) that for $a \in A$ and $r > 0$, the set $\{x \in X : \|a(x)\| \geq r\}$ be compact. It is easy to check that a function $f : X \rightarrow [0, \infty)$ is upper semicontinuous and vanishes at infinity if and only if for every $r > 0$ the set $\{x \in X : f(x) \geq r\}$ is compact.

Part (3) is now immediate from the definitions. \square

We prove results stating that if X is totally disconnected and the fibers of a $C_0(X)$ -algebra A have a particular property, then so does A . These don't require continuity. We will return to continuity later in this section, when we want to prove that if a continuous $C_0(X)$ -algebra with nonzero fibers has one of our properties, then X is totally disconnected.

Lemma 4.7. Let the notation be as in Definition 4.1 and Notation 4.2. Let $B \subset A$ be a hereditary subalgebra. Let $a \in A$. Then $a \in B$ if and only if $\text{ev}_x(a) \in \text{ev}_x(B)$ for all $x \in X$.

Proof. The forward implication is immediate.

For the reverse implication, we first claim that if $f \in C_0(X)$ satisfies $0 \leq f \leq 1$ and if $b \in B$, then $\iota(f)b \in B$. To prove the claim, it suffices to consider the case $b \geq 0$. In this case, $\iota(f)b = b^{1/2}\iota(f)b^{1/2}$, and the claim follows from the fact that B is also a hereditary subalgebra in $M(A)$.

To prove the result, it is enough to prove that for every $\varepsilon > 0$ there is $b \in B$ such that $\|a - b\| < \varepsilon$. So let $\varepsilon > 0$. Define $K \subset X$ by

$$K = \left\{ x \in X : \|\text{ev}_x(a)\| \geq \frac{\varepsilon}{2} \right\}.$$

For $x \in K$ choose $c_x \in B$ such that $\text{ev}_x(c_x) = \text{ev}_x(a)$, and define $U_x \subset X$ by

$$U_x = \left\{ y \in X : \|\text{ev}_y(c_x - a)\| < \frac{\varepsilon}{2} \right\}.$$

It follows from Lemma 4.3(2) that K is compact and from Lemma 4.3(3) that U_x is open for all $x \in K$. Choose $x_1, x_2, \dots, x_n \in K$ such that the sets $U_{x_1}, U_{x_2}, \dots, U_{x_n}$ cover K . Choose continuous functions $f_k : X \rightarrow [0, 1]$ with compact support contained in U_{x_k} for $k = 1, 2, \dots, n$, and such that for $x \in K$ we have $\sum_{k=1}^n f_k(x) = 1$ and for $x \in X \setminus K$ we have $\sum_{k=1}^n f_k(x) \leq 1$. Define $b \in A$ by

$$b = \sum_{k=1}^n \iota(f_k)c_{x_k}.$$

Then $b \in B$ by the claim. Moreover, if $x \in K$ then, using Lemma 4.3(4) at the first step, and $\|\text{ev}_x(c_{x_k} - a)\| < \frac{\varepsilon}{2}$ whenever $f_k(x) \neq 0$ at the second step, we have

$$\|\text{ev}_x(b - a)\| \leq \sum_{k=1}^n f_k(x) \|\text{ev}_x(c_{x_k} - a)\| < \frac{\varepsilon}{2}.$$

Define $f(x) = 1 - \sum_{k=1}^n f_k(x)$ for $x \in X$. For $x \in X \setminus K$, similar reasoning gives

$$\begin{aligned} \|\text{ev}_x(b - a)\| &\leq \|\text{ev}_x(b - [1 - f(x)]a)\| + \|f(x)\text{ev}_x(a)\| \\ &\leq \sum_{k=1}^n f_k(x) \|\text{ev}_x(c_{x_k} - a)\| + f(x) \|\text{ev}_x(a)\| \\ &\leq [1 - f(x)] \frac{\varepsilon}{2} + f(x) \|\text{ev}_x(a)\| \leq \frac{\varepsilon}{2}. \end{aligned}$$

It now follows from Lemma 4.3(1) that $\|b - a\| < \varepsilon$. This completes the proof. \square

Corollary 4.8. Let X be a locally compact Hausdorff space, let A be a $C_0(X)$ -algebra with structure map $\iota : C_0(X) \rightarrow Z(M(A))$, and let $B \subset A$ be a hereditary subalgebra. Then there is a homomorphism $\mu : C_0(X) \rightarrow Z(M(B))$ which makes B a $C_0(X)$ -algebra and such that for all $b \in B$ and $f \in C_0(X)$ we have $\mu(f)b = \iota(f)b$. Moreover, $B_x = \text{ev}_x(B)$ for all $x \in X$.

Proof. It follows from Lemma 4.7 that if $f \in C_0(X)$ and $b \in B$ then $\iota(f)b \in B$. For $f \in C_0(X)$ we define $T_f: B \rightarrow B$ by $T_f(b) = \iota(f)b$ for $b \in B$. It is easy to check that (T_f, T_f) is a double centralizer of B , and that $f \mapsto (T_f, T_f)$ defines a homomorphism $\mu: C_0(X) \rightarrow Z(M(B))$. Nondegeneracy of μ follows from nondegeneracy of ι . The relations $\mu(f)b = \iota(f)b$ and $B_x = \text{ev}_x(B)$ hold by construction. \square

Lemma 4.9. Let X be a locally compact Hausdorff space, let A be a $C_0(X)$ -algebra with structure map $\iota: C_0(X) \rightarrow Z(M(A))$, let $F \subset A$ be a finite set, and let $\varepsilon > 0$. Then there is $f \in C_c(X)$ such that $0 \leq f \leq 1$ and $\|\iota(f)a - a\| < \varepsilon$ for all $a \in F$.

Proof. Define $K \subset X$ by

$$K = \{x \in X : \text{there is } a \in F \text{ such that } \|\text{ev}_x(a)\| \geq \frac{\varepsilon}{3}\}.$$

It follows from Lemma 4.3(2) that K is compact. Choose $f \in C_c(X)$ such that $0 \leq f \leq 1$ and $f(x) = 1$ for all $x \in K$.

Fix $a \in F$. Let $x \in X$. If $x \in K$ then, using Lemma 4.3(4), $\|\text{ev}_x(\iota(f)a - a)\| = 0$. Otherwise, again using Lemma 4.3(4),

$$\|\text{ev}_x(\iota(f)a - a)\| \leq f(x)\|\text{ev}_x(a)\| + \|\text{ev}_x(a)\| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

Clearly $\sup_{x \in X} \|\text{ev}_x(\iota(f)a - a)\| \leq \frac{2\varepsilon}{3} < \varepsilon$. So $\|\iota(f)a - a\| < \varepsilon$ by Lemma 4.3(1). \square

Lemma 4.10. Let X be a locally compact Hausdorff space, let A be a $C_0(X)$ -algebra with structure map $\iota: C_0(X) \rightarrow Z(M(A))$, let $z \in X$, let $F \subset \text{Ker}(\text{ev}_z)$ be a finite set, and let $\varepsilon > 0$. Then there is $f \in C_c(X \setminus \{z\})$ such that $0 \leq f \leq 1$ and $\|\iota(f)a - a\| < \varepsilon$ for all $a \in F$.

Proof. The proof is essentially the same as that of Lemma 4.9. We define K as there, observe that $z \notin K$, and require that $\text{supp}(f)$, in addition to being compact, be contained in $X \setminus \{z\}$. \square

Lemma 4.11. Let X be a locally compact Hausdorff space, let A be a $C_0(X)$ -algebra with structure map $\iota: C_0(X) \rightarrow Z(M(A))$, and let $I \subset A$ be an ideal. Let $\pi: A \rightarrow A/I$ be the quotient map. Then there is a homomorphism $\mu: C_0(X) \rightarrow Z(M(A/I))$ which makes A/I a $C_0(X)$ -algebra and such that for all $a \in A$ and $f \in C_0(X)$ we have $\mu(f)\pi(a) = \pi(\iota(f)a)$. Moreover, giving I the $C_0(X)$ -algebra from Corollary 4.8, for every $x \in X$ we have $(A/I)_x \cong A_x/I_x$.

Proof. Let $\bar{\pi}: M(A) \rightarrow M(A/I)$ be the map on multiplier algebras induced by $\pi: A \rightarrow A/I$. Define $\mu = \bar{\pi} \circ \iota$. All required properties of μ are obvious except for nondegeneracy.

To prove nondegeneracy, let $b \in A/I$ and let $\varepsilon > 0$. Choose $a \in A$ such that $\pi(a) = b$. Use Lemma 4.9 to choose $f \in C_c(X)$ such that $0 \leq f \leq 1$ and $\|\iota(f)a - a\| < \varepsilon$. Then

$$\|\mu(f)b - b\| = \|\pi(\iota(f)a - a)\| < \varepsilon.$$

This completes the proof of nondegeneracy.

It remains to prove the last statement. Let $x \in X$. Let $\text{ev}_x: A \rightarrow A_x$ be as in Notation 4.2, and let $\bar{\text{ev}}_x: A/I \rightarrow (A/I)_x$ be the corresponding map with A/I in place of A . Also let $\pi_x: A_x \rightarrow A_x/I_x$ be the quotient map. Then $\pi_x \circ \text{ev}_x$ and $\bar{\text{ev}}_x \circ \pi$ are surjective, so it suffices to show that they have the same kernel.

Let $a \in A$. Suppose first $(\pi_x \circ \text{ev}_x)(a) = 0$. Let $\varepsilon > 0$; we prove that $\|(\bar{\text{ev}}_x \circ \pi)(a)\| < \varepsilon$. We have $\text{ev}_x(a) \in I_x$. So there is $b \in I$ such that $\text{ev}_x(b) = \text{ev}_x(a)$.

Then $\text{ev}_x(a-b) = 0$. So Lemma 4.10 provides $f \in C_c(X \setminus \{x\})$ such that $0 \leq f \leq 1$ and $\|\iota(f)(a-b) - (a-b)\| < \varepsilon$. By Corollary 4.8, we have $\iota(f)b \in I$. So $\pi(b) = \pi(\iota(f)b) = 0$. Thus

$$\|\mu(f)\pi(a) - \pi(a)\| = \|\pi(\iota(f)(a-b) - (a-b))\| < \varepsilon.$$

Since $\overline{\text{ev}}_x(\mu(f)\pi(a)) = 0$, it follows that $\|(\overline{\text{ev}}_x \circ \pi)(a)\| < \varepsilon$.

Now assume that $(\overline{\text{ev}}_x \circ \pi)(a) = 0$. Let $\varepsilon > 0$; we prove that $\|(\pi_x \circ \text{ev}_x)(a)\| < \varepsilon$. Apply Lemma 4.10 to the $C_0(X)$ -algebra A/I , getting $f \in C_c(X \setminus \{x\})$ such that $0 \leq f \leq 1$ and $\|\mu(f)\pi(a) - \pi(a)\| < \varepsilon$. Thus $\|\pi(\iota(f)a - a)\| < \varepsilon$. Choose $b \in I$ such that $\|[\iota(f)a - a] - b\| < \varepsilon$. It follows that

$$\|(\pi_x \circ \text{ev}_x)(\iota(f)a - a - b)\| < \varepsilon.$$

Since $\text{ev}_x(\iota(f)a) = 0$ and $(\pi_x \circ \text{ev}_x)(b) = 0$, it follows that $\|(\pi_x \circ \text{ev}_x)(a)\| < \varepsilon$, as desired. \square

The following result is closely related to [18].

Lemma 4.12. Let X be a locally compact Hausdorff space, let A be a $C_0(X)$ -algebra with structure map $\iota: C_0(X) \rightarrow Z(M(A))$, and let D be a C^* -algebra. Then there is a homomorphism $\mu: C_0(X) \rightarrow Z(M(D \otimes_{\max} A))$ which makes $D \otimes_{\max} A$ a $C_0(X)$ -algebra and such that for all $a \in A$, $d \in D$, and $f \in C_0(X)$ we have $\mu(f)(d \otimes a) = d \otimes \iota(f)a$. Moreover, for every $x \in X$ we have $(D \otimes_{\max} A)_x \cong D \otimes_{\max} A_x$.

Proof. The family

$$(X, (\text{id}_D \otimes_{\max} \pi_x: A \rightarrow A_x)_{x \in X}, D \otimes_{\max} A)$$

is a C^* -bundle in the sense of Definition 1.1 of [18]. (See (2) on page 678 of [18].)

Using exactness of the maximal tensor product and Lemma 4.10, one verifies the hypothesis of Lemma 2.3 of [18]. This lemma therefore implies that for $b \in D \otimes_{\max} A$ the function $x \mapsto \|\text{ev}_x(b)\|$ is upper semicontinuous. It is clear that for $d \in D$ and $a \in A$ the function $x \mapsto \|\text{ev}_x(d \otimes a)\|$ vanishes at infinity, and it then follows from density that for all $b \in D \otimes_{\max} A$ the function $x \mapsto \|\text{ev}_x(b)\|$ vanishes at infinity. Now apply Proposition 4.6. \square

Lemma 4.13. Let X be a totally disconnected locally compact Hausdorff space, let A be a $C_0(X)$ -algebra with structure map $\iota: C_0(X) \rightarrow Z(M(A))$, and let $x \in X$.

- (1) Let $p \in A_x$ be a projection. Then there is a projection $e \in A$ such that $\text{ev}_x(e) = p$.
- (2) Let $p \in A_x$ be an infinite projection. Then there is an infinite projection $e \in A$ such that $\text{ev}_x(e) = p$.

The use of semiprojectivity is slightly indirect, because we don't know that there is a countable neighborhood base at x .

Proof of Lemma 4.13. We prove (1). Since \mathbb{C} is semiprojective, there is $\varepsilon > 0$ such that whenever B and C are C^* -algebras, $\varphi: B \rightarrow C$ is a homomorphism, and $b \in B$ satisfies $\|b^* - b\| < \varepsilon$, $\|b^2 - b\| < \varepsilon$, and $\varphi(b)$ is a projection, then there exists a projection $e \in B$ such that $\varphi(e) = \varphi(b)$. Since ev_x is surjective, there is $a \in A$ such that $\text{ev}_x(a) = p$. By Lemma 4.3(3), there is an open set $U \subset X$ with $x \in U$ such that for all $y \in U$ we have $\|\text{ev}_y(a^* - a)\| < \frac{\varepsilon}{2}$ and $\|\text{ev}_y(a^2 - a)\| < \frac{\varepsilon}{2}$. Since X is

totally disconnected, there is a compact open set $K \subset X$ such that $x \in K \subset U$. Define $b = \iota(\chi_K)a$. Using Lemma 4.3(4), we get

$$\|\mathrm{ev}_y(b^* - b)\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|\mathrm{ev}_y(b^2 - b)\| < \frac{\varepsilon}{2}$$

when $y \in K$, and $\mathrm{ev}_y(b^* - b) = \mathrm{ev}_y(b^2 - b) = 0$ when $y \in X \setminus K$. It follows from Lemma 4.3(1) that

$$\|b^* - b\| \leq \frac{\varepsilon}{2} < \varepsilon \quad \text{and} \quad \|b^2 - b\| \leq \frac{\varepsilon}{2} < \varepsilon.$$

Now obtain e by using the choice of ε with $B = A$ and $C = A_x$.

We describe the changes needed for the proof of (2). Let T be the Toeplitz algebra, generated by an isometry s (so $s^*s = 1$ but $ss^* \neq 1$). By hypothesis, there is a homomorphism $\varphi_0: T \rightarrow A_x$ such that $\varphi_0(1) = p$ and $\varphi_0(1 - ss^*) \neq 0$. Since T is semiprojective, an argument similar to that in the proof of (2) shows that there is a homomorphism $\varphi: T \rightarrow A$ such that $\mathrm{ev}_x \circ \varphi = \varphi_0$. Set $e = \varphi(1)$. Then $\varphi(s)^*\varphi(s) = e$ and $\varphi(s)\varphi(s)^* \leq e$. We have $e - \varphi(s)\varphi(s)^* \neq 0$ because $\mathrm{ev}_x(e - \varphi(s)\varphi(s)^*) \neq 0$. So e is an infinite projection. \square

Theorem 4.14. Let X be a totally disconnected locally compact Hausdorff space and let A be a $C_0(X)$ -algebra.

- (1) Assume that A_x has residual (SP) for all $x \in X$. Then A has residual (SP).
- (2) Assume that A_x is purely infinite and has the ideal property for all $x \in X$. Then A is purely infinite and has the ideal property.
- (3) Assume that A_x has the weak ideal property for all $x \in X$. Then A has the weak ideal property.
- (4) Assume that A is separable and A_x has topological dimension zero for all $x \in X$. Then A has topological dimension zero.

Proof. We prove (1). Recall (Definition 7.1 of [29]) that a C^* -algebra D has residual (SP) if and only if D is residually hereditarily in the class \mathcal{C} of all C^* -algebras which contain a nonzero projection. (See (4) in the introduction.)

We verify the definition directly. So let $I \subset A$ be an ideal such that $A/I \neq 0$, and let $B \subset A/I$ be a nonzero hereditary subalgebra. Combining Lemma 4.11 and Corollary 4.8, we see that B is a $C_0(X)$ -algebra. Since $B \neq 0$, Lemma 4.3(1) provides $x \in X$ such that $B_x \neq 0$. Let $\overline{\mathrm{ev}}_x: A/I \rightarrow (A/I)_x$ be the map of Notation 4.2 for the $C_0(X)$ -algebra A/I . Then $B_x = \overline{\mathrm{ev}}_x(B)$ by Corollary 4.8 and $(A/I)_x \cong A_x/I_x$ by Lemma 4.11. Thus B_x is isomorphic to a nonzero hereditary subalgebra of A_x/I_x . Since A_x has residual (SP), it follows that there is a nonzero projection $p \in B_x$. Lemma 4.13(1) provides a projection $e \in B$ such that $\overline{\mathrm{ev}}_x(e) = p$. Then $e \neq 0$ since $\overline{\mathrm{ev}}_x(e) \neq 0$. We have thus verified that A has residual (SP).

We next prove (2). Let \mathcal{C} be the class of all C^* -algebras which contain an infinite projection. By the equivalence of conditions (ii) and (iv) of Proposition 2.11 of [31] (valid, as shown there, even when A is not separable), a C^* -algebra D is purely infinite and has the ideal property if and only if D is residually hereditarily in \mathcal{C} . (See (1) in the introduction.) The argument is now the same as for (1), except using Lemma 4.13(2) in place of Lemma 4.13(1).

Now we prove (3). Let \mathcal{C} be the class of all C^* -algebras B such that $K \otimes B$ contains a nonzero projection. It is shown at the beginning of the proof of

Theorem 8.5 of [29] that a C^* -algebra D has the weak ideal property if and only if D is residually hereditarily in \mathcal{C} . (See (5) in the introduction.)

We verify that A satisfies this condition. So let $I \subset A$ be an ideal such that $A/I \neq 0$, and let $B \subset A/I$ be a nonzero hereditary subalgebra. As in the proof of (1), B is a $C_0(X)$ -algebra and there is $x \in X$ such that B_x is isomorphic to a nonzero hereditary subalgebra of A_x/I_x . Therefore $K \otimes B_x$ contains a nonzero projection p . Since K is nuclear, Lemma 4.12 implies that $K \otimes B$ is a $C_0(X)$ -algebra with $(K \otimes B)_x \cong K \otimes B_x$. So Lemma 4.13(1) provides a projection $e \in K \otimes B$ such that $\overline{\text{ev}}_x(e) = p$. Then $e \neq 0$ since $\overline{\text{ev}}_x(e) \neq 0$. This shows that A is residually hereditarily in \mathcal{C} , as desired.

Finally, we prove (4). Since A is separable, by Theorem 1.10 it suffices to show that A is residually hereditarily in the class \mathcal{C} of all C^* -algebras D such that $\mathcal{O}_2 \otimes D$ contains a nonzero projection. Also, for every $x \in X$, the algebra A_x is separable. So Theorem 1.10 implies that A_x is residually hereditarily in \mathcal{C} . The proof is now the same as for (3), except using \mathcal{O}_2 in place of K . \square

We will next show that when the $C_0(X)$ -algebra is continuous, the fibers are all nonzero, and the algebra is separable, then the algebra has one of our properties if and only if all the fibers have this property and X is totally disconnected.

Separability should not be necessary.

Having nonzero fibers is necessary. The zero C^* -algebra is a $C_0(X)$ -algebra for any X , and it certainly has all our properties. For a less trivial example, let X_0 be the Cantor set, take $X = X_0 \amalg [0, 1]$, and make $C(X, \mathcal{O}_2)$ a $C(X)$ -algebra via restriction of functions in $C(X)$ to X_0 .

Continuity is necessary, at least without separability.

Example 4.15. Let X be any compact Hausdorff space. Let A be the C^* -algebra product $A = \prod_{x \in X} \mathcal{O}_2$, consisting of elements a in the set theoretic product such that $\sup_{x \in X} \|a_x\|$ is finite. Define a homomorphism $\iota: C(X) \rightarrow A$ by $\iota(f) = (f(x) \cdot 1)_{x \in X}$ for $f \in C(X)$. We show that this homomorphism makes A a $C(X)$ -algebra which is purely infinite and has the ideal property, the weak ideal property, topological dimension zero, and residual (SP). Taking $X = [0, 1]$ shows that without at least one of continuity of the $C(X)$ -algebra structure and separability, all four parts of Theorem 4.14 fail.

It is immediate to check that if B_x is an arbitrary nonzero unital C^* -algebra for $x \in X$, then the map $\iota: C(X) \rightarrow \prod_{x \in X} B_x$ given by $\iota(f) = (f(x) \cdot 1_{B_x})_{x \in X}$ for $f \in C(X)$ is a homomorphism which makes $\prod_{x \in X} B_x$ a $C(X)$ -algebra.

It is easy to check, by working independently in the factors in the product, that if B_x has real rank zero for all $x \in X$ then $\prod_{x \in X} B_x$ has real rank zero, and that if B_x is purely infinite and simple for all $x \in X$ then every nonzero projection in $\prod_{x \in X} B_x$ is properly infinite. In particular, in the case of $A = \prod_{x \in X} \mathcal{O}_2$, condition (iii) in Proposition 2.11 of [31] is satisfied (every hereditary subalgebra of A is generated as an ideal by its properly infinite projections). The implications from this condition to the others in Proposition 2.11 of [31] do not require separability. Thus, using condition (iv) there, we see that A has residual (SP). Using condition (ii) there, we see that A is purely infinite and has the ideal property. The weak ideal property is then immediate, and topological dimension zero now follows from Theorem 1.8.

The construction is easily adapted to spaces X which are only locally compact and possibly nonunital fibers.

It is possible that requiring separability and that all fibers be nonzero will force X to be totally disconnected.

Lemma 4.16. Let X be a second countable locally compact Hausdorff space, and let A be a separable continuous $C_0(X)$ -algebra such that $A_x \neq 0$ for all $x \in X$. If A has topological dimension zero then X is totally disconnected.

We assume that X is second countable because we need A to be separable in Theorem 1.9.

Proof of Lemma 4.16. We identify continuous $C_0(X)$ -algebras and continuous C^* -bundles as in Proposition 4.6. Now use Corollary 2.8 of [18] to see that $\mathcal{O}_2 \otimes A$ is a continuous $C_0(X)$ -algebra. It follows from Theorem 1.9 that $\mathcal{O}_2 \otimes A$ has the ideal property, and then from Theorem 2.1 of [26] that X is totally disconnected. \square

Theorem 4.17. Let X be a second countable locally compact Hausdorff space, and let A be a separable continuous $C_0(X)$ -algebra such that $A_x \neq 0$ for all $x \in X$.

- (1) A has residual (SP) if and only if X is totally disconnected and A_x has residual (SP) for all $x \in X$.
- (2) A is purely infinite and has the ideal property if and only if X is totally disconnected and A_x is purely infinite and has the ideal property for all $x \in X$.
- (3) A has the weak ideal property if and only if X is totally disconnected and A_x has the weak ideal property for all $x \in X$.
- (4) A has topological dimension zero if and only if X is totally disconnected and A_x has topological dimension zero for all $x \in X$.

Proof. In all four parts, the reverse implications follow from Theorem 4.14. Also, in all four parts, the fact that A_x has the appropriate property for all $x \in X$ follows from the general fact that the property passes to arbitrary quotients. See Theorem 7.4(7) of [29] for residual (SP), Theorem 6.8(7) of [29] for the combination of purely infiniteness and the ideal property, Theorem 8.5(5) of [29] for the weak ideal property, and combine Proposition 5.8 of [29] with Theorem 1.10(9) for the weak ideal property.

It remains to show that all four properties imply that X is totally disconnected. All four properties imply topological dimension zero (using Theorem 1.8 as necessary), so this follows from Lemma 4.16. \square

The proofs in this section depend on properties of projections, and so do not work for a general property defined by being residually hereditarily in an upwards directed class of C^* -algebras. However, we know of no counterexamples to either version of the following question.

Question 4.18. Let \mathcal{C} be an upwards directed class of C^* -algebras, let X be a totally disconnected locally compact space, and let A be a $C_0(X)$ -algebra such that A_x is residually hereditarily in \mathcal{C} for all $x \in X$. Does it follow that A is residually hereditarily in \mathcal{C} ? What if we assume that A is a continuous $C_0(X)$ -algebra?

5. STRONG PURE INFINITENESS FOR BUNDLES

It seems to be unknown whether $C_0(X) \otimes A$ is purely infinite when X is a locally compact Hausdorff space and A is a general purely infinite C^* -algebra, even when A is additionally assumed to be simple. (To apply Theorem 5.11 of [16], one also needs to know that A is approximately divisible.) Efforts to prove this by working locally on X seem to fail. Even in cases in which they work, such methods are messy. It therefore seems worthwhile to give the following result, which, given what is known already, has a simple proof.

Theorem 5.1. Let X be a locally compact Hausdorff space, and let A be a locally trivial $C_0(X)$ -algebra whose fibers A_x are strongly purely infinite in the sense of Definition 5.1 of [17]. Then A is strongly purely infinite.

Since X is locally compact, local triviality is equivalent to the requirement that every point $x \in X$ have a compact neighborhood L such that, using the $C(L)$ -algebra structure on $A|_L$ from Notation 4.2 and the obvious $C(L)$ -algebra structure on $C(L, A_x)$, these two algebras are isomorphic as $C(L)$ -algebras. We say in this case that $A|_L$ is trivial.

Proof of Theorem 5.1. Let $\iota: C_0(X) \rightarrow Z(M(A))$ be the structure map.

We first prove the result when X is compact, by induction on the least $n \in \mathbb{Z}_{>0}$ for which there are open sets $U_1, U_2, \dots, U_n \subset X$ which cover X and such that $A|_{\overline{U_j}}$ is trivial for $j = 1, 2, \dots, n$. If $n = 1$, there is a strongly purely infinite C^* -algebra B such that $A \cong C(X, B)$, and A is strongly purely infinite by Corollary 5.3 of [14]. Assume the result is known for some $n \in \mathbb{Z}_{>0}$, and suppose that there are open sets $U_1, U_2, \dots, U_{n+1} \subset X$ which cover X and such that $A|_{\overline{U_j}}$ is trivial for $j = 1, 2, \dots, n+1$. Define $U = \bigcup_{j=1}^n U_j$. If $X \setminus U = \emptyset$ then the induction hypothesis applies directly. Otherwise, use $X \setminus U \subset U_{n+1}$ to choose an open set $W \subset X$ such that $X \setminus U \subset W \subset \overline{W} \subset U_{n+1}$. Define $Y = X \setminus W$ and $L = \overline{W}$. Then

$$L \cup Y = X, \quad X \setminus L \subset Y, \quad Y \subset U, \quad \text{and} \quad L \subset U_{n+1}.$$

Since $L \subset \overline{U_{n+1}}$, there is a strongly purely infinite C^* -algebra B such that $A|_L \cong C(L, B)$. By definition (see Notation 4.2), there is a short exact sequence

$$0 \longrightarrow \overline{\iota(C_0(X \setminus L))A} \longrightarrow A \longrightarrow A|_L \longrightarrow 0.$$

We can identify the algebra $\overline{\iota(C_0(X \setminus L))A}$ with an ideal in $A|_Y$. Consideration of the sets $U_1 \cap Y, U_2 \cap Y, \dots, U_n \cap Y$ shows that the induction hypothesis applies to $A|_Y$, which is therefore strongly purely infinite. So $\overline{\iota(C_0(X \setminus L))A}$ is strongly purely infinite by Proposition 5.11(ii) of [17]. Also $A|_L$ is strongly purely infinite by Corollary 5.3 of [14], so A is strongly purely infinite by Theorem 1.3 of [14]. This completes the induction step, and the proof of the theorem when X is compact.

We now prove the general case. Let $(U_\lambda)_{\lambda \in \Lambda}$ be an increasing net of open subsets of X such that $\overline{U_\lambda}$ is compact for all $\lambda \in \Lambda$ and $\bigcup_{\lambda \in \Lambda} U_\lambda = X$. For $\lambda \in \Lambda$, the algebra $A|_{\overline{U_\lambda}}$ is strongly purely infinite by the case already done. So its ideal $\overline{\iota(C_0(U_\lambda))A}$ is strongly purely infinite by Proposition 5.11(ii) of [17]. Using Lemma 4.9, one checks that $A \cong \varinjlim_{\lambda \in \Lambda} \overline{\iota(C_0(U_\lambda))A}$, so A is strongly purely infinite by Proposition 5.11(iv) of [17]. \square

Lemma 5.2. Let A be a separable C^* -algebra. Then the following are equivalent:

- (1) A is purely infinite and has topological dimension zero.
- (2) A is strongly purely infinite and has the ideal property.

Proof. Condition (2) implies condition (1) because strong pure infiniteness implies pure infiniteness (Proposition 5.4 of [17]), the ideal property implies the weak ideal property, and the weak ideal property implies topological dimension zero (Theorem 1.8).

Now assume (1). Then A has the ideal property by Theorem 1.9. Apply Proposition 2.14 of [31]. \square

Corollary 5.3. Let X be a locally compact Hausdorff space, and let A be a locally trivial $C_0(X)$ -algebra whose fibers A_x are all purely infinite, separable, and have topological dimension zero. Then A is strongly purely infinite.

Proof. Lemma 5.2 implies that the fibers are all strongly purely infinite, so that Theorem 5.1 applies. \square

6. WHEN DOES THE WEAK IDEAL PROPERTY IMPLY THE IDEAL PROPERTY?

The weak ideal property seems to be the property most closely related to the ideal property which has good behavior on passing to hereditary subalgebras, fixed point algebras, and extensions. (Example 2.7 of [28] gives a separable unital C^* -algebra A with the ideal property and an action of \mathbb{Z}_2 on A such that the fixed point algebra does not have the ideal property. Example 2.8 of [28] gives a separable unital C^* -algebra A such that $M_2(A)$ has the ideal property but A does not have the ideal property. Theorem 5.1 of [22] gives an extension of separable C^* -algebras with the ideal property such that the extension does not have the ideal property.) On the other hand, the ideal property came first, and in some ways seems more natural. Accordingly, it seems interesting to find conditions under which the weak ideal property implies the ideal property. Our main result in this direction is Theorem 6.12. It covers, in particular, separable locally AH algebras in the sense of Definition 6.7(2) below. We also prove (Proposition 6.13) that the weak ideal property implies the ideal property for stable C^* -algebras with Hausdorff primitive ideal space. We give an example to show that this implication can fail for Z -stable C^* -algebras.

In the introduction, we illustrated the importance of the ideal property with several theorems in which it is a hypothesis. We start by showing that two of these results can otherwise fail: Theorem 4.1 of [21] (stable rank one for AH algebras with slow dimension growth) in Example 6.1, and Theorem 3.6 of [10] (AT structure when in addition the K -theory is torsion free) in Example 6.2. In both cases, Theorem 6.12 below implies that one can replace the ideal property with the weak ideal property.

Example 6.1. Let D be the 2^∞ UHF algebra. Then $C([0, 1]^2, D)$ is an AH algebra, even in the somewhat restrictive sense of Definition 6.7 of [21], which has no dimension growth. It follows from Proposition 5.3 of [19] that $C([0, 1]^2, D)$ does not have stable rank one. Thus, Theorem 4.1 of [21] fails without the ideal property.

Example 6.2. Let D be the 3^∞ UHF algebra, and let $X = [0, 1]^5$. Then $C(X, D)$ is an AH algebra with no dimension growth. We show that $C(X, D)$ has torsion free K -theory and is not an AT algebra. Thus, Theorem 3.6 of [10] fails without the ideal property.

We have $K_0(C(X, D)) \cong \mathbb{Z}[\frac{1}{3}]$ and $K_1(C(X, D)) = 0$. Thus $K_*(C(X, D))$ is torsion free. Since the real projective space $\mathbb{R}P^2$ is a compact 2-dimensional manifold, there is a closed subspace $Y \subset X$ such that $Y \cong \mathbb{R}P^2$. By Proposition 2.7.7 of [1], $K^0(\mathbb{R}P^2) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Therefore

$$K_0(C(Y, D)) \cong \mathbb{Z}[\frac{1}{3}] \otimes (\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}[\frac{1}{3}] \oplus \mathbb{Z}/2\mathbb{Z}.$$

Since this group has torsion, $C(Y, D)$ is not an AT algebra. Since $C(Y, D)$ is a quotient of $C(X, D)$, it follows that $C(X, D)$ is not an AT algebra.

It is convenient to work with the following class of C^* -algebras.

Notation 6.3. We denote by \mathcal{P} the class of all separable C^* -algebras for which topological dimension zero, the ideal property, and the weak ideal property are all equivalent.

That is, a separable C^* -algebra A is in \mathcal{P} exactly when either A has all of the properties topological dimension zero, the ideal property, and the weak ideal property, or none of them.

The class \mathcal{P} is not particularly interesting in its own right. (For example, all cones over nonzero C^* -algebras are in \mathcal{P} , because they have none of the three properties.) However, proving results about it will make possible a result to the effect that these properties are all equivalent for the smallest class of separable C^* -algebras which contains the separable AH algebras (as well as some others) and is closed under certain operations.

The following lemma isolates, for convenient reference, what we actually need to prove to show that a separable C^* -algebra is in \mathcal{P} .

Lemma 6.4. Let A be a separable C^* -algebra for which topological dimension zero implies the ideal property. Then $A \in \mathcal{P}$.

Proof. The ideal property implies the weak ideal property by Proposition 8.2 of [29]. The weak ideal property implies topological dimension zero by Theorem 1.8. \square

We prove two closure property for the class \mathcal{P} . What can be done here is limited by the failure of other closure properties for the class of C^* -algebras with the ideal property. See the introduction to this section. (It is hopeless to try to prove results for \mathcal{P} for quotients, since the cone over every C^* -algebra is in \mathcal{P}).

Lemma 6.5. Let $(A_\lambda)_{\lambda \in \Lambda}$ be a countable family of C^* -algebras in \mathcal{P} . Then $\bigoplus_{\lambda \in \Lambda} A_\lambda \in \mathcal{P}$.

Proof. Set $A = \bigoplus_{\lambda \in \Lambda} A_\lambda$. Then A is separable, since Λ is countable and A_λ is separable for all $\lambda \in \Lambda$. By Lemma 6.4, we need to show that if A has topological dimension zero then A has the ideal property. For $\lambda \in \Lambda$, the algebra A_λ is a quotient of A , so has topological dimension zero by Proposition 2.6 of [5] and Lemma 3.6 of [28]. Therefore A_λ has the ideal property by hypothesis.

It is clear that arbitrary direct sums of C^* -algebras with the ideal property also have the ideal property, so it follows that A has the ideal property. \square

Lemma 6.6. Let A and B be C^* -algebras in \mathcal{P} . Assume that A is exact. Then $A \otimes_{\min} B \in \mathcal{P}$.

Proof. The algebra $A \otimes_{\min} B$ is separable because A and B are. By Lemma 6.4, we need to show that if $A \otimes_{\min} B$ has topological dimension zero then $A \otimes_{\min} B$ has the ideal property. Now A and B have topological dimension zero by Theorem 3.6, so have the ideal property by hypothesis. It now follows from Corollary 1.3 of [30] that $A \otimes_{\min} B$ has the ideal property. \square

We now identify a basic collection of C^* -algebras in \mathcal{P} . The main point of the first class we consider is that it contains the separable AH algebras (as described below), but in fact it is much larger.

Since there are conflicting definitions of AH algebras in the literature, we include a definition. Our version is more restrictive than some other versions in the literature, because we insist on spaces with only finitely many connected components and direct systems with injective maps. We don't need to use direct sums in the definition because we don't assume that the projections involved have constant rank.

Definition 6.7. Let A be a C^* -algebra.

- (1) We say that A is an *AH algebra* if A is a direct limit of a sequence $(A_n)_{n \in \mathbb{Z}_{\geq 0}}$ of C^* -algebras of the form $pC(X, M_k)p$ for a compact Hausdorff space X with finitely many connected components, $k \in \mathbb{Z}_{>0}$, and a projection $p \in C(X, M_k)$, all depending on n , and in which the maps $A_n \rightarrow A_{n+1}$ are all injective.
- (2) We say that A is a *locally AH algebra* if for every finite set $F \subset A$ and every $\varepsilon > 0$, there exist a compact Hausdorff space X with finitely many connected components, $k \in \mathbb{Z}_{>0}$, a projection $p \in C(X, M_k)$, and an injective homomorphism $\varphi: pC(X, M_k)p \rightarrow A$ such that for all $a \in F$ there is $b \in pC(X, M_k)p$ with $\|\varphi(b) - a\| < \varepsilon$.

In particular, AH algebras are locally AH algebras.

Definition 6.8. Let A be a C^* -algebra.

- (1) We say that A is *standard* (Definition 2.7 of [6]) if A is unital and whenever B is a simple unital C^* -algebra and $J \subset A \otimes_{\min} B$ is an ideal which is generated as an ideal by its projections, then there is an ideal $I \subset A$ which is generated as an ideal by its projections and such that $J = I \otimes_{\min} B$.
- (2) We say that A is an *LS algebra* (Definition 2.13 of [6]) if for every finite set $F \subset A$ and every $\varepsilon > 0$, there exist a standard C^* -algebra D and an injective homomorphism $\varphi: D \rightarrow A$ such that for all $a \in F$ there is $b \in D$ with $\|\varphi(b) - a\| < \varepsilon$.

Lemma 6.9. Let A be a C^* -algebra.

- (1) If $A \cong pC(X, M_k)p$ for a compact Hausdorff space X with only finitely many connected components, $k \in \mathbb{Z}_{>0}$, and a projection $p \in C(X, M_k)$, then A is standard.
- (2) If A is a locally AH algebra, then A is an LS algebra.

Proof. Part (1) is a special case of Remark 2.9(2) of [6]. Part (2) is immediate from part (1). \square

There are many more standard C^* -algebras than in Lemma 6.9(1), and therefore many more LS algebras than in Lemma 6.9(2). For example, in Definition 6.7(2) replace $pC(X, M_k)p$ by a finite direct sum of C^* -algebras of the form $pC(X, D)p$

for connected compact Hausdorff spaces X , simple unital C^* -algebras D , and projections $p \in C(X, D)$. Such a C^* -algebra is standard by Remark 2.9(2) of [6], so a direct limit of a system of such algebras with injective maps is an LS algebra. (When all the algebras D which occur are exact and the direct system is countable, but the maps of the system are not necessarily injective, such a direct limit is called an exceptional GAH algebra in [24]. See Definitions 2.9 and 2.7 there.)

Lemma 6.10. Let A be a separable LS algebra (Definition 6.8(2)). Then $A \in \mathcal{P}$.

Proof. As usual, we use Lemma 6.4. Assume A has topological dimension zero. By the implication from (1) to (3) in Theorem 1.10, the algebra $\mathcal{O}_2 \otimes A$ has the ideal property. Apply Lemma 2.11 of [6] with $B = \mathcal{O}_2$ to conclude that A has the ideal property. \square

Extending the list of properties in the discussion of type I C^* -algebras in Remark 2.12 of [26] (and using essentially the same proof as there), we get the following longer list of equivalent conditions on a separable type I C^* -algebra.

Proposition 6.11. Let A be a separable type I C^* -algebra. Then the following are equivalent:

- (1) A has topological dimension zero.
- (2) A has the weak ideal property.
- (3) A has the ideal property.
- (4) A has the projection property (every ideal in A has an increasing approximate identity consisting of projections; Definition 1 of [23]).
- (5) A has real rank zero.
- (6) A is an AF algebra.

Proof. It is clear that every condition on the list implies the previous one. So we need only show that (1) implies (6). Use Lemma 3.6 of [28] to see that $\text{Prim}(A)$ has a base for its topology consisting of compact open sets. Then the theorem in Section 7 of [3] implies that A is AF. \square

Theorem 6.12. Let \mathcal{W} the smallest class of separable C^* -algebras which contains the separable LS algebras (including the separable locally AH algebras), the separable type I C^* -algebras, and the separable purely infinite C^* -algebras, and is closed under finite and countable direct sums and under minimal tensor products when one tensor factor is exact. Then for any C^* -algebra in \mathcal{W} , topological dimension zero, the weak ideal property, and the ideal property are all equivalent.

Proof. Combine Lemma 6.5, Lemma 6.6, Lemma 6.10, Lemma 6.9(2), Proposition 6.11, and Theorem 1.9. \square

Proposition 6.13. Let A be a C^* -algebra such that $\text{Prim}(A)$ is Hausdorff. If A has the weak ideal property then $K \otimes A$ has the ideal property.

In particular, the weak ideal property implies the ideal property for stable C^* -algebras with Hausdorff primitive ideal space.

Proof of Proposition 6.13. Arguing as in the proof of Proposition 3.13, we see that $K \otimes A$ is a $C_0(\text{Prim}(A))$ -algebra, with fibers $(K \otimes A)_P \cong K \otimes (A/P)$ for $P \in \text{Prim}(A)$. Moreover, $\text{Prim}(A)$ is totally disconnected, and for every $P \in X$, the quotient A/P is simple and has the weak ideal property.

For $P \in \text{Prim}(A)$, it follows that $K \otimes (A/P)$ is simple and has a nonzero projection, so has the ideal property. This is true for all $P \in \text{Prim}(A)$, so $K \otimes A$ has the ideal property by Theorem 2.1 of [26]. \square

Let Z be the Jiang-Su algebra. It is unfortunately not true that the weak ideal property implies the ideal property for Z -stable C^* -algebras.

Example 6.14. We give a separable C^* -algebra A such that A and $Z \otimes A$ have the weak ideal property but such that neither A nor $Z \otimes A$ has the ideal property.

Let D be a Bunce-Deddens algebra, and let the extension

$$0 \longrightarrow K \otimes D \longrightarrow A \longrightarrow \mathbb{C} \longrightarrow 0$$

be as in the proof of Theorem 5.1 of [22]. (The extension is as in the first paragraph of that proof, using the choices suggested in the second paragraph.) In particular, A does not have the ideal property, and the connecting homomorphism $\exp: K_0(\mathbb{C}) \rightarrow K_1(K \otimes D)$ is injective. Since $K \otimes D$ and \mathbb{C} have the weak ideal property (for trivial reasons), it follows from Theorem 8.5(5) of [29] that A has the weak ideal property. Clearly $Z \otimes K \otimes D$ and $Z \otimes \mathbb{C}$ have the ideal property. However, it is shown in the proof of Theorem 2.9 of [27] that $Z \otimes A$ does not have the ideal property.

Question 6.15. Let A be a separable C^* -algebra which is a direct limit of recursive subhomogeneous C^* -algebras. If A has the weak ideal property, does A have the ideal property?

We suspect that the answer is no, but we don't have a counterexample.

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